A unifying definition of synchronization for dynamical systems

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We propose a unifying definition for synchronization between stationary finite dimensional deterministic dynamical systems. By example, we show that the synchronization phenomena discussed in the dynamical systems literature fits within the framework of this definition, and discuss problems with previous definitions of synchronization. We conclude with a discussion of possible extensions of the definition to infinite dimensional systems described by partial differential equations and/or systems where noise is present. © 2000 American Institute of Physics.

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Synchronization between coupled systems is a topic whose roots go back to (at least) Huygens. Over the last decade synchronization between nonlinear and/or spatially extended systems has been extensively studied, and many “new” types of synchronization have been announced. At this time there is no definition of synchronization that consistently covers the known examples of this phenomena. This paper attempts to fill this void by proposing a unified definition of synchronization that is accessible to most scientific researchers. Examples are used to demonstrate applications of the proposed definition and to explain why other definitions fail. The definition offers a common language and structure that can be used for future research in the area of synchronization.

I. INTRODUCTION

Synchronization between dynamical systems has been an active research topic since the time of Huygens. It is a phenomenon of interest to fields ranging from celestial mechanics to laser physics, and from communication to neuroscience.1

Over the last decade, a number of new types of synchronization have appeared: chaotic synchronization,2 phase synchronization,3 lag synchronization,4 and generalized synchronization,5 to mention only a few. This is in addition to the classic examples of synchronization in periodic systems.6,7 Furthermore, many of these behaviors have been experimentally observed in a single system.8

Synchronization is often categorized on the basis of whether the coupling mechanism is unidirectional or bidirectional. Stable synchronization with unidirectional coupling has been called synchronization by an external force (for frequency synchronization) and master-slave synchronization (Pecora and Carroll5). However, it has recently been shown that, if the synchronized systems are identical then there is no essential difference between unidirectional and bidirectional synchronization.9

Although there have been several attempts,10,11 currently no successful unified definition of synchronization exists. Indeed, it seems that the definition in use is an enumerated list. When a “new type” of synchronization arises, its name is added to the list. We believe that “definition by example” is an untidy situation which should be replaced by a single definition that encompasses all of the known examples.

In this short paper we propose a unified definition which covers for all types of synchronization between stationary deterministic finite dimensional systems. Although we explicitly discuss synchronization between two continuous time dynamical systems, our results cover N continuous time and N discrete time systems. Therefore, our results apply to a larger class of phenomena than that one we explicitly discussed.

At the end of this paper we discuss possible extensions of the definition to stationary infinite dimensional systems. Such systems can be either deterministic partial differential equations or nominally finite dimensional systems where the presence of noise is considered. Synchronization in such systems has received increasing attention in the literature.

II. CONSTRUCTING THE DEFINITION

To construct the definition, assume that a large stationary deterministic finite dimensional dynamical system is divided into two subsystems,

\[
\frac{dx}{dt} = f_1(x,y;t), \quad \frac{dy}{dt} = f_2(y,x;t). \tag{1}
\]

Here, \(x \in \mathbb{R}^{d_1}\) and \(y \in \mathbb{R}^{d_2}\) are vectors which may have different dimensions. The phase space and vector field of the large system is formed (in a natural way) from the product of the two smaller phase spaces and vector fields. Examples of phenomena that can be described by Eq. (1) are ubiquitous.

Colloquially, synchronization means correlated in-time behavior between different processes. Indeed, the Oxford Advanced dictionary,12 defines synchronization as “to agree
in time’ and ‘to happen at the same time.’ From this intuitive definition we propose that synchronization requires the following four tasks:

1. Separating the dynamics of a large dynamical system into the dynamics of subsystems.
2. Measuring properties of the subsystems.
3. Comparing properties of the subsystems.
4. Determining whether the properties agree in time.

If the properties agree then the systems are synchronized. The remainder of this paper formalizes this intuitive definition of synchronization by explicitly addressing each requirement, and applying the proposed definition to examples.

We begin by separating the dynamics of the large systems into the dynamics of subsystems. Let \( \mathcal{X} \) denote a trajectory of the large dynamical system, given by Eq. (1), with the initial condition, \( z_0 = [x_0, y_0] \in \mathbb{R}^4 \otimes \mathbb{R}^2 \). Respectively, curves \( \phi_i(z_0) \) and \( \phi_j(z_0) \) are obtained from this trajectory by projecting away the \( y \) and \( x \) components. We say that \( \phi_i(z_0) \) and \( \phi_j(z_0) \) are ‘trajectories’ of the first and second subsystems of Eq. (1). In this context we have separated the trajectories \( \phi_i(z_0) \) and \( \phi_j(z_0) \) from \( \phi_i(z_0) \), rather than constructing \( \phi(z_0) \) from \( \phi_i(z_0) \) and \( \phi_j(z_0) \).

We now discuss measuring properties of the subsystems. Let \( \mathcal{X} \) denote the space of all trajectories of the first subsystem, and consider a function \( g : \mathcal{X} \otimes \mathbb{R} \rightarrow \mathbb{R} \) which is not identically zero.\(^1\) The first \( R \) represents time, and is included so that \( g \) may make explicit reference to time. We say that the function, \( g, \) is a property of the first subsystem. The image of \( \{ \phi_i(z_0), i \} \in \mathcal{X} \otimes \mathbb{R} \) under \( g \) is the result of measuring the property of the first subsystem, and will be denoted by \( g(x) \in \mathbb{R}^k \). Similar definitions can be made for the second subsystem. The following examples make these notions less abstract.

Frequency is a subsystem property that is often of interest in synchronization research. Notice that the property, frequency, is not the same as measuring the property. Measuring the property means calculating a numerical value for the frequency. Hence, \( g \) is the thing being measured, while \( g(x) = \omega \) is the value of the measurement. Also, frequency is a time average over the full trajectory of the sub-system. Therefore, in principle, one needs all of \( \phi_i(z_0) \in \mathcal{X} \) to calculate the scalar \( \omega \). Finally, notice that the property, \( g, \) and its measured value, \( g(x) \), only make explicit reference to the trajectory of the \( x \) subsystem. The nature and number of the other subsystems are not explicitly referenced. The ability to independently and unambiguously measure properties of a subsystem by considering only trajectories of the subsystems arises from the separation of the large dynamical system into subsystems. This illustrates the need for the separation step in our discussion of synchronization.

Another property which is often of interest in synchronization is the phase space coordinates of the subsystem at time \( t \). Here, the property is the phase space coordinates of the subsystem, while measuring the property means determining numerical values for the coordinates. Experimentally, \( g \) is the property being measured, and \( g(x) = x(t) \) are the values of the measurement. For this example (unlike the previous one) the property is a \( d_1 \)-dimensional vector, not a scalar. Also, the property implicitly depends on time, and is not a time average. Therefore, it must be obtained from \( \mathcal{X} \otimes \mathbb{R} \) because \( \phi_i(z_0) \in \mathcal{X}, \) alone, is not sufficient for measuring the property.

These examples show that the generality of our definition of property as a function from \( \mathcal{X} \otimes \mathbb{R} \rightarrow \mathbb{R}^k \) cannot be avoided because synchronization involves measuring time averages for some examples, and properties that depend implicitly on time for other examples. Furthermore, \( k, \) the dimension of the measurement, can take on different values depending on what is being measured. The key idea that \( g \) represents a property of the \( x \) subsystem is enforced by the fact that \( g(x) \) can be obtained without explicitly referring to the number or nature of the other subsystems.

Finally, we discuss the notions of comparing the properties, and determining when they agree in time. We say the time independent function \( h : \mathbb{R}^k \otimes \mathbb{R} \rightarrow \mathbb{R} \) compares the measured properties of the two subsystems, and the two measurements agree in time if and only if \( h[g(x), g(y)] = 0 \). Below, a norm is used to determine this last requirement.

With these preliminaries in place, we offer the following definition for synchronization:

**Definition:** The sub-systems in Eqs. (1) are synchronized on the trajectory \( \phi(z_0) \), with respect to the properties \( g \) and \( g \), if there is a time independent mapping \( h : \mathbb{R}^k \otimes \mathbb{R} \rightarrow \mathbb{R} \) such that

\[
\| h[g(x), g(y)] \| = 0.
\]

Here, \( \| \cdot \| \) is any norm.

A subsequent definition removes details of initial conditions and trajectories.

**Definition:** The subsystems in Eqs. (1) are synchronized with respect to the properties \( g \) and \( g \) if there is a time independent mapping \( h : \mathbb{R}^k \otimes \mathbb{R} \rightarrow \mathbb{R} \) such that

\[
\| h[g(x), g(y)] \| = 0,
\]

holds on all trajectories.

This subsequent definition is what many papers in the literature call synchronization.\(^1\) However, as we and others have shown, synchronization depends strongly on the trajectory.\(^1,14\) Two subsystems can be synchronized on some trajectories and not synchronized on other trajectories. Therefore, the trajectory dependence in the first definition cannot be ignored.

The stability of synchronous motion is another issue raised by these two definitions. Specifically, stability is not required by the first definition. This definition only requires that \( h = 0 \) exists for properties measured on the trajectory. If the trajectory of one of the sub-systems is perturbed then this condition may no longer hold. As a trivial example of this, consider two uncoupled identical Lorenz systems with parameter values that produce chaotic trajectories. If both systems have the same initial condition then they will follow the same trajectory and their motion will clearly be synchronous. However, because of chaos, this type of synchronization is very unstable. In contrast, the second definition implicitly requires the notion of stability because it requires \( h = 0 \) for all trajectories.
We claim that these definitions naturally follow from the intuitive definition of synchronization given above, and that they encompass all of the interesting examples found in the literature.

A strength of the definition is that the properties and comparison functions are not specified, \textit{a priori}. As the examples below show, different applications require different properties and comparison functions. Those that are suitable for one application are often completely unsuitable for another. For example, the following comparison functions all appear in the literature:

\begin{align*}
h[g(x), g(y)] &= g(x) - g(y), \\
h[g(x), g(y)] &= \lim_{t \to \infty} [g(x) - g(y)], \\
h[g(x), g(y)] &= \lim_{T \to \infty} \frac{1}{T} \int_{t}^{t+T} [g(x(s)) - g(y(s))] ds.
\end{align*}

Similar breadth occurs with properties. For example, the literature offers examples where the same two subsystems are synchronized with respect to some properties, and are not synchronized with respect to other properties.

A lack of breadth is the Achilles heel of previous definitions. Typically, they fail because they, \textit{a priori}, specify properties and/or comparison functions that must be applied to all types of synchronization. As we show below, this makes it easy to find examples where the selected property and/or comparison is inappropriate, and two sub-systems that are said to be synchronized by the research community are not synchronized under the definition.

Some may be concerned that the definition is too general. We argue, via analogy, that this argument lacks weight. Perhaps the most useful concept in theoretical physics is a vector space. The definition of a vector space is as general as the one proposed for synchronization.\textsuperscript{15} The definition does not specify what constitutes a "vector" or the operation "+," Thus, a range of things from matrices, to Fourier series, to bras and kets are vectors in their respective vector spaces. The definition only insists that the set of "vectors" obey a specific series of abstract rules. If a set obeys these rules then it is a vector space, and the considerable power one obtains from that knowledge can be employed. (Group theory is another example of an extremely useful concept in physics whose definition is abstract.) Our definition, gives an explicit list of four tasks that must be satisfied before one can say synchronization occurs. Like the definition of a vector space (or a group) it provides a structural framework that can be used for subsequent research. We believe that this structure is an improvement over the current situation of definitions that have numerous counterexamples and/or enumerated lists.

In the next section we demonstrates the utility of the definition by applying it to well-known examples.

III. EXAMPLES

In this section we discuss examples of synchronization found in the literature.

A. Frequency synchronization

The subsystem properties used in frequency synchronization are frequencies. If the trajectory is mostly rotation about an axis then the measured frequencies \([\omega_z = g(x)]\) and \([\omega_z = g(y)]\) are located at power spectra peaks associated with the average rotation of the signal. Examples of such dynamics include, periodic motion, systems with phase coherent chaotic attractors (like Rössler\textsuperscript{16}), or systems with Silnikov dynamics.\textsuperscript{17} For these examples, phase modulation contributes weakly to the dynamics, and chaos (if it exists) results mainly from amplitude modulation.

The measurement function is typically \(h[g(x), g(y)] = n_x \omega_z - n_y \omega_y\), (where \(n_x\) and \(n_y\) are integers). Synchronization implies that the frequencies of the sub-systems are commensurate

\[n_x \omega_z - n_y \omega_y = 0.\]

Many textbooks discuss frequency synchronization for periodic systems.\textsuperscript{7}

Frequency synchronization between coupled chaotic systems with bistable attractors has also been examined.\textsuperscript{18} (The Lorenz and double scroll attractors are examples of bistable attractors.) In Ref. 18, the properties are the average frequency of switching between the two lobes of the attractors, and frequency synchronization on a trajectory occurs if Eq. (5) is satisfied.

Our definition works for this example, while many other definitions fail. For example, any definitions that use phase space coordinates as the measured property and requires \(\|x-y\|=0\) will fail because \(\|x-y\|\) need not remain small when the subsystems are frequency synchronized.\textsuperscript{10}

Another example is frequency synchronization between a chaotic bistable attractor and a periodic system. The communication method of Hayes \textit{et al.}\textsuperscript{19} labels the lobes of the attractor as 0 or 1, and uses small control signals to produce a trajectory which encodes the message. The obvious choice for the sampling rate of the receiver is the mean switching frequency of the chaotic system. Therefore, a periodic receiver with frequency \(\omega_y\) is synchronized to a chaotic transmitter with switching frequency \(\omega_x\) if \(\omega_x = \omega_y\). This type of synchronization is covered by our definition, but does not seem to fit into any previous definition.

Frequency synchronization compares properties that are time averages of the trajectory. Therefore, it is a loose restriction on the dynamics of the subsystems. In particular, it does not restrict the instantaneous values of the coordinates \(x\) and \(y\). All remaining examples compare properties whose measured values depend implicitly on time, and restrict the instantaneous values of \(x\) and \(y\).

B. Phase synchronization

Phase synchronization involves subsystem properties called "phases." If the dynamics is chaotic and phase coherent, then one can introduce cylindrical coordinates, and unambiguously define the phase as the angle coordinate, \(\phi(t)\). However, other applications define the phase via a Hilbert transform, in which case the phase may not be uniquely defined on the subsystem.\textsuperscript{3} Also, there are examples where the
measured phase, \( \phi(t) \), is a vector obtained from a trajectory using none of the previous methods.\textsuperscript{20} Therefore, any definition of synchronization that uses a fixed definition for the property “phase” will fail because phase is ambiguous. Therefore, the property that is measured will likely differ from one researcher to the next.

If the measured properties are given by \( \mathbf{g}(x) = \phi_x(t) \) and \( \mathbf{g}(y) = \phi_y(t) \) then the most common comparison mapping is\textsuperscript{3,4,21}

\[
\mathbf{h}[\mathbf{g}(x), \mathbf{g}(y)] = U[\epsilon(\mathbf{g}(x) - \mathbf{g}(y))].
\]

Here, \( U(u,v) \) is a vector with \( \alpha \)th component \( U_\alpha(u,v) = \Theta[|u_\alpha| - |v_\alpha|] \), where \( \Theta \) is the unit step function. The comparison function in Eq. (6) implies that subsystems are synchronized if \( |\phi_x - \phi_y| < \epsilon \). Here, \( \epsilon \) is the maximum tolerable separation between the components of the phase. The value of \( \|\epsilon\| \) is usually small, but can not be set \textit{a priori} because its size is application dependent.\textsuperscript{21} If “phase slips” occur then a comparison mapping which uses a time average [see Eq. (4)] is necessary.\textsuperscript{3}

Phase synchronization only compares the phase variables. In the synchronous state the phases are locked, but the amplitudes can remain chaotic and relatively uncorrelated. Our definition easily includes phase synchronization. In contrast, definitions which focus on \( \|x - y\| \), or a specific notions of phase, or insist on a particular measurement function will fail for reasons just given.

\section{C. Identical synchronization}

This is the most frequently discussed form of synchronization within the nonlinear dynamics community.\textsuperscript{1} Here the subsystems are identical, and the properties are the phase space variables, \( \mathbf{g}(x) = x(t) \), and \( \mathbf{g}(y) = y(t) \). Most discussions in the literature use Eq. (3) as the comparison function.\textsuperscript{1,10}

However, if the dynamics of the subsystems are chaotic then bursts (sudden loss and recovery of synchronous motion caused by unstable periodic orbits within the attractor) occur on chaotic trajectories for some forms of coupling.\textsuperscript{14} Applications which cannot tolerate bursts demand high quality synchronization, where Eq. (6) is used as the comparison function.\textsuperscript{21} If bursts are tolerable then a hybrid of Eqs. (4) and (6) can be used.

An engineering application called “dead-beat” synchronization (only possible in discrete time dynamical systems) uses Eq. (2) as the comparison function.\textsuperscript{22} This type of synchronization is often used to describe systems whose measured properties are restricted to a finite symbolic alphabet.\textsuperscript{23}

In their seminal paper, Afraimovich, Verichev, and Rabinovich\textsuperscript{24} generalized identical synchronization in two different ways, one is now called lag synchronization and the other is now called generalized synchronization.

\section{D. Lag synchronization}

Two subsystems are lag synchronized if their measured properties lag each other by a fixed amount of time, \( \tau \). A trivial example is when the measured properties are \( \mathbf{g}(x) = x(t) \) and \( \mathbf{g}(y) = y(t+\tau) \), and the comparison function is Eq. (2). For this example, the \( y \) subsystem follows the same trajectory as the \( x \) subsystem, but is \( \tau \) units of time behind.

A nontrivial example is given in Ref. 4, where the measured properties are \( \mathbf{g}(x) = x_1(t) \) and \( \mathbf{g}(y) = y_1(t+\tau) \) (the first components of \( x \) and \( y \)). The comparison function is

\[
\mathbf{h}[\mathbf{g}(x), \mathbf{g}(y)] = \lim_{i \to \infty} \frac{1}{T} \int_{t}^{t+T} [\mathbf{g}(x(s)) - \mathbf{g}(y(s))]^2 ds,
\]

where \( K \) is a constant. For this example, the subsystems are not identical and \( S^2(\tau) = \|\mathbf{h}\| = 0 \) for a nonzero value of \( \tau \).

Lag synchronization also occurs if, instead of a constant value of \( \tau \), one uses \( \mathbf{g}(y) = y(\tau) \), with \( T : \mathbb{R} \to \mathbb{R} \) a homeomorphism with \( \lim_{\tau \to \infty} [T(\tau)/\tau] = 1.24 \).

The second generalization in Ref. 24 follows from the observation: If the subsystems are identical, then the set \( x = y \) defines an invariant manifold in the phase space of the large system.

\section{E. Generalized synchronization}

Roughly speaking, generalized synchronization examines synchronization between sub-systems whose vector fields are not identical because they have different functional form and/or different parameter values. It is also possible for the subsystems to have different dimensions (\( d_1 \neq d_2 \)). The literature is not consistent when discussing generalized synchronization. For example, most papers say that generalized synchronization occurs if the measured properties are \( \mathbf{g}(x) = x \) and \( \mathbf{g}(y) = y \), and the comparison function is given by

\[
\mathbf{h}[\mathbf{g}(x), \mathbf{g}(y)] = \mathbf{H}[\mathbf{g}(x)] - \mathbf{g}(y),
\]

where \( \mathbf{H} \) is a smooth, invertible, time independent function.\textsuperscript{11} Roughly speaking, the sub-systems are generally synchronized if \( y(t) = \mathbf{H}[x(t)] \). More importantly, the equation, \( y = \mathbf{H}(x) \), defines an invariant manifold (in the phase space of the large system) which can be used to determine the state of one subsystem given the state of the other subsystem.\textsuperscript{11} If the subsystems are identical then \( \mathbf{H}(x) = x \).

However, Rulkov \textit{et al.}\textsuperscript{5} discuss an example of generalized synchronization where the vector fields of the subsystems have the same functional form but different parameter values. Their numerical and experimental evidence indicates that it is possible to have stable frequency synchronization on a trajectory and not have generalized synchronization in the sense discussed above. In their example, one subsystem oscillated twice for every oscillation of the other subsystem (i.e., \( \omega_1/\omega_2 = 2 \)). This implies that it is not possible to construct a smooth invertible function \( \mathbf{H} \) such that \( y = \mathbf{H}(x) \). Therefore, the definitions in Ref. 11 fail.

Rulkov \textit{et al.}’s example illustrates that a definition of generalized synchronization needs to include \( \mathbf{H}’s \) with a finite (or perhaps countable) number of branches. Similar conclusions arise from the work of Pyragas.\textsuperscript{25} This work examined generalized synchronization using two identical coupled Logistic maps as subsystems and the phase space coordinate as the measured property. Using the auxiliary system method, the author shows that the subsystems exhibit generalized synchronization prior to identical synchronization, as the coupling strength is increased. More importantly, for this
example the measurement function for generalized synchronization could not be written as \( h(x,y) = y - H(x) \) with smooth invertible \( H \).

Our definition only requires a time independent functional relationship, \( h(g(x), g(y)) = 0 \), between the measured properties of the subsystem. In particular, we do not insist that this relationship can be rewritten as \( g(y) = H[g(x)] \), where \( H \) is smooth and invertible. Thus, the examples discussed in Refs. 5 and 25 easily fit into our definition. In fact, how one might extend our definition to these systems we discuss an example examined by Neiman et al. and others view as synchronization between the stochastic systems. Here Eqs. (1) are replaced by coupled partial differential equations (PDE’s). For these systems the “state” is an infinite dimensional field. As such, \( \chi \) in the domain of \( g_1 \) and \( g_2 \) will have to change to an infinite dimensional space composed of “trajectories of fields.” In addition, the range of these functions will need to include the possibility of functions instead of just vectors. These changes imply that the range and domain of the function \( h \) will also have to become to infinite dimensional function space and the norm used to determine \( \|h\| = 0 \) will have to be extended to include norms on functions. In our opinion, none of these changes is difficult (in principle), and we do not foresee major difficulties in extending our definition to these systems. (Of course it is possible that difficulties that we do not foresee will arise which make it impossible to perform this extension.)

The second class contains stochastic systems. Here Eqs. (1) are replaced by differential equations with stochastic terms. These systems have been the focus of intensive research in dynamics community. Much of this work has focused on stochastic resonance, which the authors of Ref. 27 and others view as synchronization between the stochastic input and the state of the bistable system. To demonstrate how one might extend our definition to these systems we discuss an example examined by Neiman et al.27

The example is an intrinsically noisy symmetric bistable system driven by inputs that stochastically switch between two possible values. The authors define the “states of the system” as \( \sigma, d \), where \( \sigma = \pm 1 \) are the two possible outputs, and \( d = \pm 1 \) are the two possible inputs of the system. The dynamics of this four state Markov model is governed by the following master PDE for \( P_{\sigma,d} = P(\sigma,d,t|\sigma_0,d_0,t_0) \), the conditional probability density of the system,

\[
\partial_t P_{\sigma,d} = -(W_{\sigma,-\sigma}(d) + \gamma)P_{\sigma,d} + \gamma P_{\sigma,-d} + W_{-\sigma,d}(d)P_{-\sigma,d}
\]

Here \( W_{\sigma,-\sigma}(d) \) and \( W_{-\sigma,d}(d) \) are functions of \( d \) which represent transition rates between the two output states, and \( \gamma \) is the mean flipping rate of the input signal.

We now discuss the four steps of our formalism in the context of this example. The first step is separating the large dynamical system. To do so we must to rewrite Eq. (8) as four coupled PDE’s of the form,

\[
\partial_t P_\sigma = F_\sigma(P_\sigma, P_d), \quad \partial_t P_d = F_d(P_d, P_\sigma),
\]

where \( P_\sigma = P_{\sigma,1} + P_{\sigma,-1} \) and \( P_d = P_{d,1} + P_{d,-1} \). Assuming this can be done the equations of motion for the large dynamical system has been rewritten as equations for two subsystems, one with fields \( P_\sigma \) and one with fields \( P_d \). Furthermore, the function space “trajectory” of the separate fields can be obtained by solving the coupled equations.

In our formalism \( \sigma \) and \( d \) are examples of properties \( g_1 \) and \( g_2 \). Since the fields \( P_\sigma \) and \( P_d \) are probabilities it is natural to define \( g(x) \), the measured value of the property \( g_1 \), as the expectation value corresponding to \( g_2 \). [For example, \( \sigma(t) = (g_1(t) = \langle g_2(\sigma_1^n(t)) \rangle \).] By defining \( g_1 \) as the process of obtaining an expectation value we insure that \( g_1 \) is a function acting on a “trajectory” of \( P_\sigma \) and possibly time.

In their work Neiman et al. advocate measuring, \( \langle T \rangle = \langle 1 \rangle \_ \) the mean duration of the output states, \( \pm 1 \). In principle, these quantities are measured properties of the output sub-system which can be directly calculated as expectation values over \( P_\sigma \). From this, \( \langle T \rangle = \langle 1 \rangle \_ + \langle 1 \rangle \_ \), the mean period of switching between the output states can be obtained. Measuring the corresponding properties of the input system are also possible, and will render \( 2/\gamma \), by definition.

The comparison function they use is \( h = \pi l(T) \_ - \pi g \), and the norm is any scalar norm. The meaning of this comparison function is clear if one defines \( \langle T \rangle = 2 \pi /\langle T \rangle \_ \) and \( \langle 1 \rangle \_ \) as the mean switching frequencies of the output and inputs subsystems, respectively. The authors report a range of values for the size of the intrinsic noise where frequency synchronization occurs, \( \|h\| = 0 \) which implies that \( \langle 0 \rangle = \langle 1 \rangle \_ \). Neiman et al. also discuss phase synchronization for a stochastic system described by

\[
\frac{dx}{dt} = x - x^3 + Qd(t) + \sqrt{2D} \xi(t),
\]

where \( d(t) \) is the same type of dichotomic noise described above, and \( \xi(t) \) is intrinsic thermal noise. The parameters \( Q \) and \( D \) define the size of the noise.

We believe that our definition can be extended to this type of system by adding an equation describing the noise processes, \( d \). This will be an equation for the time evolution of the probability \( P_d \). One can view this extended set of equations as a large system that has been separated into subsystems (one for the dynamics of \( x \) and \( \xi \), and another for the dynamics of \( d \)). For this example the authors are interested in synchronization between \( x \) and \( d \), so the dynamics of \( \xi \) need not be explicitly separated. Since the dynamics of both \( x \) and \( d \) are stochastic their “trajectory” will have to be described in terms of probability densities instead of curves in a Euclidean space.

For this example, the process of measuring a property is equivalent to obtaining an expectation value. The measure-
ment function used is the same one discussed above for phase synchronization. The authors report a value of $D$ that leads to $|\mathbf{h}| = 0$ which implies phase synchronization between $x$ and $d$ this stochastic system.

As a final word about stochastic dynamical systems, we believe that our definition will typically force the use of an expectation value for the measuring properties of a subsystem. To understand this belief consider a system with Gaussian noise. For this case a noise disturbance of arbitrarily large size will eventually occur. When this occurs a measured property that is not an expectation value will experience a sudden change and it is unlikely that $|\mathbf{h}|$ will remain zero. Therefore, in the absence of an expectation value for the property it is unlikely that the subsystems will remain synchronized under our (or anyone else) definition of synchronization.

V. CONCLUSIONS

In conclusion, we have describe four tasks that are required for synchronization. Based on this, we proposed a unified definition of synchronization between stationary finite dimensional dynamical systems. We claim that this definition encompasses all examples of synchronization discussed in the literature, and that it offers a common language and framework that can be used to discuss different types of synchronization.

At this point we do not claim that this definition can be extended to infinite dimensional system. However, we present possible extensions of the definition to these types of systems. Although we do not claim to have performed a comprehensive study of this topic, the major difference between finite and infinite dimensional systems seems to involve possible changes in the structure of the domain and range of the properties and measurement functions, and the use of expectation values for the process of measuring a property for stochastic systems.

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7See the discussion of Arnold tongues in E. Ott, Chaos in Dynamical Systems (Cambridge University Press, New York, 1994).
13The requirement that $g_x$ is nonzero means that the property one chooses to measure can not be zero for all possible trajectories of the subsystem and all times. This restriction on $g_x$ eliminates the trivial case where everything is synchronized to everything when one chooses to measure and compare nothing.