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The Averaging Method and the Persistence of Attractors

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Abstract The theorems that are presented in this paper, are a contribution to the foundations of the averaging method for ordinary differential equations. They involve the study of the persistent features of vector fields, under non autonomous perturbations of mean value zero. The problem of obtaining qualitative information from the study of the averaged equation is considered and theorems that give new conditions to guarantee the uniform validity of the approximation over the time interval $[0,\infty)$, are proved. A general result on the persistence of attractors is presented. The analysis uses in a fundamental way, a generalization of the notion of a solution stable under persistent disturbances. The proofs do not require special behavior of the linearized system and the results obtained are not only local, but give relevant information about the persistence of domains of attraction.

KEY WORDS: Averaging, persistency, attractors total stability, uniform asymptotic stability, stability under persistent disturbances.

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1. THE AVERAGING METHOD

Let us consider a differential equation of the form

(1) $$\dot{x} = f(t/\varepsilon,x,\varepsilon)$$

where the function $f(t,x,\varepsilon)$ is almost periodic in t, uniformly with respect to (x,ε) in compact sets and ε is a small positive parameter. Along with this equation we consider the following equation:

(2) $$\dot{\bar{x}} = f_o(\bar{x}) = \lim_{T\to\infty} (1/T)\int_0^T f(t,\bar{x},0)dt.$$

This equations is obtained from (1) by averaging its right hand side with respect to the variable t and is called the *averaged equation*. The averaging method consists of approximating the solutions of the equation (1) by the solutions of the equation (2).

The intuition behind this approximation is that, for ε small, the equation (1) corresponds to a t-dependent vector field that undergoes very rapid oscillations as t changes; it is then natural to expect that in the first approximation the solutions of (1) will only obey the average effect of the vector field $f(t,x,0)$. In the case where $f(t,x,\varepsilon)$ is periodic in t, the validity of this reasoning is easily justified by the fact that the difference between the integral of a periodic function and its mean value tends to zero as the period tends to zero.

Using a different time scale $(\tau = t/\varepsilon)$ in the equation (1), one obtains the equation

$$\text{(E)} \qquad \dot{x} = \varepsilon f(t,x,\varepsilon),$$

where the new time τ have been renamed t again. Doing the same with the equation (2) the averaged equation takes the form:

$$\text{(AE)} \qquad \dot{\bar{x}} = \varepsilon f_o(\bar{x}).$$

The study of equations in the form (E) was initiated by N.N. Bogolyubov [1] and were called by him equations in *standard form*. In spite of the inconvenience of working with an ε-dependent averaged equation, it has become a tradition in the literature to formulate and discuss the results on the foundations of the averaging method for equations in this form. Following this custom we will work in this paper with equations in standard form, assuming that in the equation (E) the function $f: \mathbb{R} \times \mathbb{C}^n \times (0,\infty) \to \mathbb{C}^n$ is continuous, has continuous partial derivate with respect to x and is almost periodic in t, uniformly with respect to (x,ε) in compact sets. Here $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ the set of complex numbers.

A fundamental result of the averaging theory is the Bogolyubov-Hale *decomposition theorem* [2]. It says that after a change of variables, the equation (E) is just a perturbation of the averaged equation. In view of this theorem the averaging procedure can be treated in many respects as a classical regular perturbation problem.

Theorem 1.

Given any compact set $\Omega \subset \mathbb{C}^n$ *there is an* $\varepsilon_o > 0$ *and a function* $u(t,y,\varepsilon)$ *continuous on* $\mathbb{R} \times \mathbb{C}^n \times (0,\varepsilon_o]$ *such that:*

i) $u(t,y,\varepsilon)$ *is almost periodic in* t *uniformly with respect to* y *in compact sets for each fixed* ε;

ii) $u(t,y,\varepsilon)$ *has a continuous derivative with respect to* t *and derivatives of any arbitrary specified order with respect to* y;

iii) $\varepsilon u(t,y,\varepsilon)$, *and* $\varepsilon \frac{\partial u}{\partial y}(t,y,\varepsilon)$ *tend to zero as* $\varepsilon \to 0$ *uniformly with respect to* $t \in \mathbb{R}$ *and* y *in compact sets;*

iv) *The change of variables*

$$x = \begin{cases} y + \varepsilon u(t,y,\varepsilon) & \text{for } (t,y,\varepsilon) \in \mathbb{R} \times \Omega \times (0,\varepsilon_o] \\ y & \text{for } (t,y,\varepsilon) \in \mathbb{R} \times \Omega \times \{0\} \end{cases} \tag{3}$$

transforms equation (E) *into*

$$\dot{y} = \varepsilon f_o(y) + \varepsilon g(t,y,\varepsilon)$$

This function, $g(t,y,\varepsilon)$, *is continuous, has continuous derivative with respect to* y *on* $\mathbb{R} \times \Omega \times [0,\varepsilon_o)$ *and approaches zero as* $\varepsilon \to 0$ *uniformly with respect to* $t \in \mathbb{R}$ *and* y *in compact sets.*

2. VALIDITY OF THE AVERAGING METHOD OVER COMPACT INTERVALS OF TIME

We recall here a theorem that implies that, for all solutions, the averaging approximation as $\varepsilon \to 0$, is uniformly valid during finite intervals of time. It is an immediate consequence of the decomposition theorem and the continuity of solutions with respect to initial conditions and parameters [3].

Theorem 2.

Let $\phi_\varepsilon(t)$ *be a solution of the equation* (AE), *defined for all* $t \geq 0$. *Given a tolerance* $\eta > 0$ *and* T *as large as we please, there are positive numbers* $\varepsilon_o(\eta,T)$ *and* $\delta(\eta,T)$ *such that:* $0 < \varepsilon < \varepsilon_o$ *and*

$$|x_o - \phi_\varepsilon(0)| < \delta$$

implies that

$$|x_\varepsilon(t,0,x_o) - \phi_\varepsilon(t)| < \eta \quad \textit{for all} \quad t \in [0,T/\varepsilon].$$

Here $x_\varepsilon(t,0,x_o)$ *is the solution of the equation* (E) *that a time* $t = 0$ *takes the value* x_o.

This theorem is a more general versión of the so called: *Bogolyubov's theorem on the validity of the averaging method over large intervals of time* [1]. It is more general because it is not restricted to compare solutions that start with the same initial conditions but neighboring initial conditions are allowed.

3. VALIDITY OF THE AVERAGING METHOD OVER UNBOUNDED TIME INTERVALS

Theorem 2 justifies the application of the averaging procedure, for any arbitrary solution and during intervals of time as long as we want, by taking ε small enough. In spite of this the approximation can break down in the long run regardless on how small we choose the value of ε to be [3]. This is a serious problem from the point of view of comparing the asymptotic behavior of the solutions of the equation (E), with that of its averaged equation.

Tne problem of finding conditions under which the approximation is valid over infinite intervals of time has been studied by several authors [4],[5],[6],[7]. It has been found that it is valid to make this approximation for solutions that have strong stability properties.

In this paper we will be interested in the following problem:

Suppose that it is known that the set A is an attractor for the averaged equation, with certain domain of attraction $\mathcal{D}(A)$; What can be said about the qualitative behavior of the solutions of the original equation? Is there a set that is attracting them? What is the domain of attraction of this set? In Section 5 we will present and prove some results on this problem. These are general results that give new conditions under which the averaging approximation provides meaningful information and contain, as particular cases, the results that the authors mentioned above have obtained for strongly stable solutions.

4. ATTRACTORS

In what follows we will let $\mathbb{R}^+$ denote the set of all non negative real numbers. The distance between a point $x \in \mathbb{C}^n$ and a set $Y \subset \mathbb{C}^n$ will be denoted by

$$d(x,Y) = \inf\{|x - y| \mid y \in Y\}.$$

Let us consider $A \subset \mathbb{R}^+ \times \mathbb{C}^n$ such that for all $t \geq 0$, the set

$$A(t) = \{x \in \mathbb{C}^n \mid (t,x) \in A\},$$

is bounded. We will say that the set A is an *attractor* for the differential equation

$$\dot{x} = f(t,x), \tag{6}$$

if it is a uniformly asymptotically stable set for it, that is:

(i) Given $\eta > 0$ there is a number $\delta > 0$ such that, for all $t_o \geq 0$,

$$d(x_o, A(t_o)) < \delta$$

implies that

$$d(x(t,t_o,x_o),A(t)) < \eta \quad \text{for all} \quad t \geq t_o,$$

where $x(t,t_o,x_o)$ denotes the solution of the equation (6) that satisfies the initial condition $x(t,t_o,x_o) = x_o$.

(ii) There exists a positive number h such that, for any given $\eta > 0$ there is T such that, for any $t_o \geq 0$,

$$d(x_o,A(t_o)) < h$$

implies that

$$d(x(t,t_o,x_o),A(t)) < \eta \quad \text{for} \quad t \geq t_o + T.$$

A solution ϕ of the equation (6) with positive semiorbit,

$$\gamma^+ = \{x \in \mathbb{C}^n \mid x = \phi(t) \quad \text{with} \quad t \geq 0\},$$

is called *uniformly orbitally asymptotically stable if the set* $\mathbb{R}^+ \times \gamma^+$ is an attractor. It is called *uniformly asymptotically stable* when its graph for $t \geq 0$ is an attractor.

The *domain of attraction* of an attractor A is the set

$$\mathcal{D}(A) = \{(t_o,x_o) \in \mathbb{R}^+ \times \mathbb{C}^n \mid d(x(t,t_o,x_o),A(t)) \to 0 \quad \text{as} \quad t \to \infty\}.$$

For every attractor this set is an open subset of $\mathbb{R}^+ \times \mathbb{C}^n$. If the equation is autonomous, $\dot{x} = f(x)$, there is a set $D \subset \mathbb{C}^n$ such that $\mathcal{D}(A) = \mathbb{R}^+ \times D$. The proof of this last assertion is trivial when the attractor is of the form $\mathbb{R}^+ \times A$ with A a subset of $\mathbb{C}^n$; for the general case, however, it requires a more elaborate argument [11].

The appendix contains other basic facts and remarks, about attractors and asymptotic stability of sets.

5. PERSISTENCE OF ATTRACTORS

Let A be an attractor of the averaged equation $\dot{\bar{x}} = f_o(\bar{x})$, with domain of attraction $\mathcal{D}(A) = \mathbb{R}^+ \times D$. Let $x_\varepsilon(t,t_o,x_o)$ be the general solution of the equation (E). We say that A *attracts solutions of the equation* (E) *in the limit* $\varepsilon \to 0$, if the set

$$A_\varepsilon = \{(t,x) \in \mathbb{R}^+ \times \mathbb{C}^n \mid (\varepsilon t, x) \in A\}$$

satisfies:

(i) Given $\eta > 0$ there are positive numbers $\varepsilon_o(\eta)$ and $\delta(\eta)$, such that:

$$(t_o,\varepsilon) \in \mathbb{R}^+ \times (0,\varepsilon_o] \quad \text{and} \quad d(x_o, A_\varepsilon(t_o)) < \delta$$

implies that

$$d(x_\varepsilon(t,t_o,x_o), A_\varepsilon(t)) < \eta \quad \text{for} \quad t \in [t_o,\infty);$$

(ii) Given any $x_o \in D$ and $\eta > 0$, there is a positive number $\varepsilon_o(x_o,\eta)$ and a function $T:(0,\varepsilon_o) \to \mathbb{R}^+$ such that for $0 < \varepsilon < \varepsilon_o$,

$$d(x_\varepsilon(t,0,x_o), A_\varepsilon(t)) < \eta \quad \text{for} \quad t \in [T(\varepsilon),\infty).$$

We now state and prove the main result of this paper.

Theorem 3.
Every attractor of the averaged equation $\dot{\bar{x}} = f_o(\bar{x})$ *attracts solutions of the equation* (E) *in the limit* $\varepsilon \to 0$.

Proof.
(A) The notion of a solution stable under persistent disturbances (totally stable) can be generalized for more general sets in the following way:

Consider the equation

(7) $\dot{x} = f(t,x).$

A set $M \subset \mathbb{R}^+ \times \mathbb{C}^n$ will be called *stable under persistent disturbances* for the equation (7) if, for any given $\eta > 0$, there are positive numbers $\delta_1(\eta)$ and $\delta_2(\eta)$, such that for any function g, the solutions of the equation

(8) $\dot{y} = g(t,y)$

have the property that, for any $t_o \geq 0$, $d(y_o,M(t_o)) < \delta_1$ and $|f(t,y) - g(t,y)| < \delta_2$, for all (t,y) such that $t \geq t_o$ and $d(y,M(t)) < \eta$, implies that

$$d(y(t,t_o,y_o)),M(t)) < \eta \quad \text{for} \quad t \geq t_o.$$

Here $y(t,t_o,y_o)$ is a solution of the equation (8).

The proof of Malkin's theorem that Yoshizawa gives in reference [8] can be easily adapted to prove the following result.

Lemma

Every attractor is stable under persistent disturbances.

(B) Notice that A_ε is an attractor for the equation (AE) with domain of attraction equal to the domain of attraction of the attractor A. Let us denote by $x_\varepsilon(t,t_o,x_o)$ the general solution of the equation (E). We will prove first that:

Given $\eta > 0$ there are $\varepsilon_o(\eta) > 0$ and $\delta > 0$ such that: $(t_o,\varepsilon) \in \mathbb{R}^+ \times \mathbb{R}^+ \times (0,\varepsilon_o)$ and $d(x_o,A_\varepsilon(t_o)) < \delta$, implies that

$$d(x_\varepsilon(t,t_o),A_\varepsilon(t)) < \eta \quad \text{for} \quad t \geq t_o.$$

To do this, let Ω be equal to the closure of the set of $x \in \mathbb{C}^n$, such that there is $(t,y) \in A$ satisfying $d(x,y) < 1$. We apply the Theorem 1 to guarantee that there are ε_1 and a function $u(t,x,\varepsilon)$ such that the change of variables $x = y + \varepsilon u(t,y,\varepsilon)$, with $0 < \varepsilon < \varepsilon_1$, transforms the equation (E) into

(Y) $\dot{y} = \varepsilon f_o(y) + \varepsilon F(t,y,\varepsilon)$

Rescaling time $(\tau = \varepsilon t)$ we obtain

(Z) $$dz/d\tau = f_o(z) + F(\tau/\varepsilon,z,\varepsilon).$$

Let y_ε and z_ε denote the solutions of the equations (Y) and (Z), respectively. Assuming that a value $\eta < 1$ is given, we choose η_o to be a positive number smaller than η. By Malkin's lemma, the attractor A is stable under persistent disturbances for the equation $z' = f_o(z)$. Let us consider $\delta_1(\eta_o)$, $\delta_2(\eta_o)$ satisfying the definition of stability under persistent disturbances and ε_2 such that $|F(t,z,\varepsilon)| < \varepsilon_2$ for (t,z) in the neighborhood of A of radius η and ε in $(0,\varepsilon_2)$. From the uniqueness of the solutions with respect to initial conditions we have that $y_\varepsilon(t,t_o,y_o) = z_\varepsilon(\varepsilon t,\varepsilon t_o,y_o)$. Therefore, if $0 < \varepsilon < \varepsilon_2$ and $d(y_o,A_\varepsilon(t_o)) < \delta_1$, then

(9) $$d(y_\varepsilon(t,t_o,y_o),A_\varepsilon(t)) = d(z_\varepsilon(\varepsilon t,\varepsilon t_o,y_\varepsilon),A_\varepsilon(t)) < \eta_o$$

for $t \geq t_o$.

Choose $\delta < \delta_1$ and let r and ε_3 be such that:

(a) $\Omega \subset B_r = \{x \in \mathbb{C}^n \mid |x| < r\}$;

(b) $|\varepsilon u(t,x,\varepsilon)| < \min\{\eta - \eta_0,\ \delta_1 - \delta\}$ and $|\varepsilon \frac{\partial u}{\partial x}(t,x,\varepsilon)| < \frac{1}{2}$

for $(t,x,\varepsilon) \in B_{r+\delta_1} \times (0,\varepsilon_o)$.

Choose $\varepsilon_o = \min\{\varepsilon_1,\varepsilon_2,\varepsilon_3,1\}$ and consider the map $G_{t,\varepsilon}(y) = x_o - \varepsilon u(t,y,\varepsilon)$, where $(t_o,t,\varepsilon) \in \mathbb{R}^+ \times \mathbb{R} \times (0,\varepsilon_o)$ and $d(x_o,A_\varepsilon(t_o)) < \delta$. Let

$$X = \{x \in \mathbb{C}^n \mid d(x,A_\varepsilon(t_o)) < \delta_1\}.$$

For any y, $y' \in X$ this map satisfies:

$$d(G_{t,\varepsilon}(y),A_\varepsilon(t_o)) \leq d(x_o-\varepsilon u(t,y,\varepsilon),x_o)+d(x_o,A_\varepsilon(t_o))<\delta_1-\delta+\delta=\delta_1$$

and

$$|G(y)-G(y')|=\varepsilon|u(t,y,\varepsilon)-u(t,y',\varepsilon)|=\varepsilon|g(1)-g(0)|=\varepsilon|\int_0^1 g'(s)ds|$$

$$=\varepsilon|\int_0^1\{\frac{\partial u}{\partial x}(t,y'+s(y-y'),\varepsilon)\}(y-y')ds|<\frac{1}{2}|y-y'|,$$

where $g(s) = u(t,y'+s(y-y'),\varepsilon)$. This implies that $G_{t,\varepsilon}:X \to X$ is a contraction.

Therefore, given $(t_o,\varepsilon) \in \mathbb{R}^+ \times (0,\varepsilon_o)$ and x_o such that $d(x_o,A_\varepsilon(t_o)) < \delta$, there is a y_o such that $x_o = y_o + \varepsilon u(t,y_o,\varepsilon)$ and $d(y_o,A_\varepsilon(t_o)) < \delta_1$. From (9) we have that

$$d(y_\varepsilon(t,t_o,y_o),A_\varepsilon(t)) < \eta_o \quad \text{for} \quad t \geq t_o.$$

Then by the uniqueness of the solutions with respect to initial conditions we have that

$$d(x_\varepsilon(t,t_o,x_o),A_\varepsilon(t)) = d(y_\varepsilon(t,t_o,y_o)+\varepsilon u(t,y_\varepsilon(t,t_o,y_o),\varepsilon),A_\varepsilon(t))$$

$$\leq d(y_\varepsilon(t,t_o,y_o),A_\varepsilon(t)) + |\varepsilon u(t,y_\varepsilon(t,t_o,y_o)|$$

$$< \eta_o + \eta - \eta_o = \eta, \quad \text{for} \quad t \geq t_o.$$

(C) Assuming that $\mathbb{R}^+ \times D$ is the domain of attraction of the attractor A, and that $x_o \in D$ and $\eta > 0$ are given, we prove that there are positive numbers ε_o and T such that, if $0 < \varepsilon < \varepsilon_o$, then

$$d(x_\varepsilon(t,0,x_o),A_\varepsilon(t)) < \eta \quad \text{for} \quad t \geq t_o + T/\varepsilon.$$

Denote by $\bar{x}_\varepsilon$ the solutions of the equation (AE). According to what was proved in part (B), we can choose δ and ε_1 such that $d(\xi_o,A_\varepsilon(t_o)) < \delta$ and $0 < \varepsilon < \varepsilon_1$, implies that

$$d(x_\varepsilon(t,t_o,\xi_o),A_\varepsilon(t)) < \eta \quad \text{for} \quad t \geq t_o.$$

Since x_o is in the domain of attraction of A, there is $T > 0$

such that

$$d(\bar{x}_1(t,0,x_o),A(t)) < \delta/2 \quad \text{for} \quad t \geq T.$$

From this it follows that

$$d(\bar{x}_\varepsilon(t,0,x_o),A_\varepsilon(t)) < \delta/2 \quad \text{for} \quad t \geq T/\varepsilon.$$

Let ε_2 be such that

$$|x_\varepsilon(t,0,x_o)-\bar{x}_\varepsilon(t,0,x_o)| < \delta/2 \quad \text{for} \quad t\in[0,T/\varepsilon] \text{ and } 0 < \varepsilon < \varepsilon_2.$$

Then if we let $\varepsilon_o = \min\{\varepsilon_1,\varepsilon_2\}$, it follows that for $0 < \varepsilon < \varepsilon_o$,

$$\begin{aligned} d(x_\varepsilon(T/\varepsilon,0,x_o),A_\varepsilon(T/\varepsilon)) &\leq |x_\varepsilon(T/\varepsilon,0,x_o) - \bar{x}_\varepsilon(T/\varepsilon,0,x_o)| + \\ &\quad d(\bar{x}_\varepsilon(T/\varepsilon,0,x_o),A_\varepsilon(T/\varepsilon)) \\ &< \delta/2 + \delta/2 = \delta. \end{aligned}$$

Therefore $d(x_\varepsilon(t,0,x_o),A_\varepsilon(t)) < \eta$ for $t \geq T/\varepsilon$. This concludes the proof of Theorem 3.

The following theorem gives conditions under which the averaging approximation is uniformly valid over the interval of time $[0,\infty)$. Part (a) focuses on the validity of the approximations for solutions and part (b) on the validity of the approximations for orbits. The problem of the validity in an orbital sense is very important because interesting solutions, like the periodic solutions, can be orbitally stable but can never be asymptotically stable.

Theorem 4.

Let ϕ *be a bounded solution, on* $\mathbb{R}^+$, *of the averaged equation* (AE).

(a) *If* ϕ *is uniformly asymptotically stable, with domain of attraction* $\mathbb{R}^+ \times D$, *then for any* $x_o \in D$,

$$|x_\varepsilon(t,0,x_o) - \bar{x}_\varepsilon(t,0,x_o)| \to 0$$

as $\varepsilon \to 0$, *uniformly with respect to* t *in* $[0,\infty)$.

(b) *If* ϕ *is uniformly orbitally asymptotically stable, with domain of attraction* $\mathbb{R}^+ \times D$, *then for any* $x_o \in D$,

$$d(x_\varepsilon(t,0,x_o),\gamma^+(\bar{x}_\varepsilon(t,0,x_o))) \to 0$$

as $\varepsilon \to 0$, *uniformly with respect to* t *in* $[0,\infty)$.

Here, x_ε *and* $\bar{x}_\varepsilon$ *are solutions of the equations* (E) *and* (AE), *respectively and* $\gamma^+(\bar{x}_\varepsilon(t,0,x_o))$ *is the positive semiorbit of the solution* $\bar{x}_\varepsilon(t,0,x_o)$.

Proof.

From Theorem 2.(a) of the appendix, we have that the set

$$A = \{(t,x) \in \mathbb{R}^+ \times \mathbb{C}^n | x = \bar{x}_1(t,0,x_o)\}$$

is an attractor for the equation $\dot{\bar{x}} = f_o(\bar{x})$. Then it follows from Theorem 3 that A attracts solutions of the equation (E) in the limit $\varepsilon \to 0$. Hence, given any $\eta > 0$ there is an $\varepsilon_o > 0$ such that, for any ε in $[0,\varepsilon_o)$,

$$|x_\varepsilon(t,0,x_o) - \bar{x}_\varepsilon(t,0,x_o)| = d(x_\varepsilon(t,0,x_o),A_\varepsilon(t)) < \eta$$

for $t \in [0,\infty)$.

This proves part (a). Analogously, to prove part (b), we use Theorem 2.(b) of the appendix to conclude that the set

$$A = \mathbb{R}^+ \times \gamma^+(\bar{x}_1(t,0,x_o)),$$

is an attractor for the equation $\dot{\bar{x}} = f_o(\bar{x})$. Then, Theorem 3 implies that A attracts solutions of the equation (E) in the limit $\varepsilon \to 0$. From this we have that, given any $\eta > 0$, there is an ε_o such that, for ε in $[0,\varepsilon_o)$,

$$d(x_\varepsilon(t,0,x_o),\gamma^+(\bar{x}_\varepsilon(t,0,x_o))) = d(x_\varepsilon(t,0,x_o),A_\varepsilon(t)) < \eta,$$

for $t \in (0,\infty)$.

6. EXAMPLES AND REMARKS

Theorem 4 applies in particular to the case of exponentially asymptotically stable equilibrium solutions, which is a well known result [7]. Notice that it also applies to any asymptotically stable equilibrium (not necessarily exponentially attracting) and to non static solutions like stable limit cycles (see appendix). For the case of a limit cycle, since exponential attraction is not necessary, the result applies even when the periodic orbit has a non trivial center manifold. Similarly, for the case of attracting manifolds, Theorem 3 does not require them to be normally hyperbolic.

Example

Let us consider, three coupled second order differential equations of the kind that correspond to one degree of freedom mechanical systems:

$$\ddot{x} + \mu_1(t/\varepsilon)(x^2-1)\dot{x} + \omega_1(t/\varepsilon)x = F_1(x,\dot{x},y,\dot{y},z,\dot{z},\varepsilon) + G_1(t/\varepsilon,x,\dot{x},y,\dot{y},z,\dot{z},\varepsilon)$$

$$(10)\quad \ddot{y} + \mu_2(t/\varepsilon)(y^2-1)\dot{y} + \omega_2(t/\varepsilon)y = F_2(x,\dot{x},y,\dot{y},z,\dot{z},\varepsilon) + G_2(t/\varepsilon,x,\dot{x},y,\dot{y},z,\dot{z},\varepsilon)$$

$$\ddot{z} + f(t/\varepsilon,z,\varepsilon)\dot{z}^3 + g(t/\varepsilon,z,\varepsilon) = F_3(x,\dot{x},y,\dot{y},z,\dot{z},\varepsilon) + G_3(t/\varepsilon,x,\dot{x},y,\dot{y},z,\dot{z},\varepsilon).$$

Two of them are van der Pol equations and the third is a pendulum like equation, with a nonlinear friction term. We assume that the system is weakly coupled for small ε. We also assume that there is non autonomous coupling through terms that are highly oscillatory in t. For these terms we do not assume the amplitude vanishes in the limit $\varepsilon \to 0$, but that they are oscillating with mean value zero. Also, other rapidly oscillating terms and coefficients, appear.

Precisely we assume that: the functions f, g, F_i and G_i are of class $\mathbb{C}^1$; the functions f, g and G_i are almost periodic in their first variable, uniformly with respect to the other variables in compact sets; the functions F_i vanish when $\varepsilon = 0$; the mean value with respect to t of the functions $f(t,z,0)$ and $g(t,z,0)$ are $f_o(z)$ and $g_o(z)$, respectively; $f_o(z)$ is a positive function and $g_o(z)$ satisfies $zg_o(z) > 0$, for $z \neq 0$; the functions μ_i and ω_i are almost periodic with positive mean values $\bar{\mu}_i$ and $\bar{\omega}_i$, respectively.

Rescaling time $(\tau = t/\varepsilon)$ and writing system (10) as a first order system we obtain a system in standard form. Clearly, the corresponding averaged system,

$$
\begin{aligned}
dx_1/d\tau &= \varepsilon x_2 \\
dx_2/d\tau &= -\varepsilon\mu_1(x_1^2 - 1)x_2 - \varepsilon\bar{\omega}_1 x_1 \\
dy_1/d\tau &= \varepsilon y_2 \\
dy_2/d\tau &= -\varepsilon\mu_2(y_1^2 - 1)y_2 - \varepsilon\bar{\omega}_2 y_1 \\
dz_1/d\tau &= \varepsilon z_2 \\
dz_2/d\tau &= -\varepsilon f_o(z_1)z_2^3 - \varepsilon g_o(z_1),
\end{aligned}
\tag{11}
$$

has an invariant 2-torus, which we will call, T. All points of $\mathbb{R}^6$, except those laying on some subsets of measure zero, approach T as $t \to \infty$. These subsets correspond to other invariant sets or the domain of attraction of them, for appropiate restrictions of the flow. Note that these other invariant sets include equilibria and periodic orbits but none of them are attractors for the full averaged equation. Also, note that the nonlinear friction term in the pendulum equation causes this torus not to be normally hyperbolic.

Being asymptotically stable, it follows from Theorem 1 of the appendix, that the set $\mathbb{R}^+ \times T$ is, according to our definition, an attractor for the averaged system (11). Then, Theorem 3 can be applied to conclude that the set $\mathbb{R}^+ \times T$ attracts solutions of the first order system corresponding to system (10), in the limit $\varepsilon \to 0$. Thus, we have that, for almost all initial conditions in $\mathbb{R}^6$, the long term evolution of this non autonomous first order system will be confined to a small neighborhood of the torus, T.

7. APPENDIX. Asymptotically stable sets and attractors.

The concept of attracting set that we are using here corresponds to what is referred to in the literature as a uniformly asymptotically stable set. There is another weaker notion of attracting set that has been widely used in the literature. According to this notion, a set $A \subset \mathbb{R}^+ \times \mathbb{C}^n$ is called an attractor for the differential equation

$$\dot{x} = f(t,x), \tag{12}$$

if it is an asymptotically stable set for it, that is:

(i) Given any cylinder around A, there is a neighborhood of A that under the flow, in the space $\mathbb{R}^+ \times \mathbb{C}^n$, remains in the prescribed cylinder. Here, by a cylinder of radius η around the set A, we mean the set of all points (t,x) such that $d(x,A(t)) < \eta$.

(ii) There is a neigborhood of A such that the orbits through points of it approach the set A, as t tends to infinity.

Clearly, the weakness of this concept of attractor is in its lack of robustness (stability) with respect to perturbations of the vector field. Sets that satisfy only the property (i) have been also called attractors in the literature. Asymptotically stable sets and attractors in this latter sense do not have the interesting persistence properties that we have discussed before.

Obviously, uniform asymptotic stability implies asymptotic stability, and the converse is not true. There exist several techniques to prove that a set is asymptotically stable e.g.: eigenvalues analysis or Lyapounov's theory for general nonlinear systems. To verify uniform asymptotic stability is more difficult in general, but in some cases it follows from asymptotic stability. The following proposition [9] gives conditions under which this is true.

Theorem 1.
Assume that in the equation (12) *the function* f *satisfies* $f(t+T,x) = f(t,x)$ *for all* t *and* x. *Then, every asymptotically stable set* A *such that* $A(t+T) = A(t)$ *for all* $t \geq 0$, *is an attractor.*

Notice that this theorem implies that if ϕ is an orbitally asymptotically stable periodic solution of an autonomous equation with orbit $\gamma(\phi)$, then the set $\mathbb{R}^+ \times \gamma(\phi)$ is an attractor. It also implies that the graphs of constant asymptotically stable solutions of autonomous equations are attractors. However, is not true that for any solution, ϕ, of an autonomous equation, to be asymptotically stable, implies that the graph of ϕ is an attractor [10].

The fact of A being an attractor does not imply that the solutions in its domain of attraction ought to be asymptotically stable. Furthermore, the solutions in its domain of attraction could even be orbitally unstable [11]. The next theorem is about some particular cases of interest, in which the attractor determines some kind of attracting behavior of the solutions on its domain of attraction.

Theorem 2.
Assume that A *is an attractor for the differential equation* (12). *Then for any* $(t_0,x_0) \in \mathcal{D}(A)$:
(a) *If* A *is the graph for* $t \geq 0$ *of a solution, then the graph*

of the solution $x(t,t_o,x_o)$, *for* $t > 0$, *is an attractor.*

(b) *If* A *is of the form* $\mathbb{R}^+ \times \gamma^+(\phi)$, *where* $\gamma^+(\phi)$ *denotes the positive semiorbit of a solution* ϕ *of the equation* (12), *then the set* $\mathbb{R}^+ \times \gamma^+(x(t,t_o,x_o))$ *is an attractor.*

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