PERTURBATION RESULTS ON THE LONG RUN BEHAVIOR OF NONLINEAR DYNAMICAL SYSTEMS

Humberto Carrillo Calvet*

ABSTRACT. The validity of the averaging approximations for ordinary differential equations in standard form is discussed. New results on the uniform validity over infinite intervals of time are presented.

1. INTRODUCTION. In this paper we will be concerned with differential equations of the form

(E) $$\dot{x} = \varepsilon\, f(t,x,\varepsilon)$$

where ε is a small positive parameter. The function f is going to be considered oscillatory with respect to the variable t in a very general sense that includes periodicity as a particular case. Precisely, we will assume that $f: \mathbf{R} \times \mathbf{R}^n \times (0,\infty) \to \mathbf{R}^n$ is continuous, has a continuous partial derivative with respect to x and is almost periodic in t, uniformly with respect to (x,ε) in compact sets. Equations like (E) were first studied by N. Krylov and N. N. Bogolyubov who called them differential equations in *standard form*.

There is a wide spectrum of dynamical systems models where such equations appear. They are very important in Mechanics and in more general systems that present nonlinear oscillation phenomena. For instance, the equation

(1) $$\dot{x} = f(x) + g(t/\varepsilon)$$

with $f: \mathcal{D} \subset \mathbf{R}^n \to \mathbf{R}^n$ and g a periodic vector valued function, models an autonomous dynamical system being acted on by a rapidly oscillating external forcing. Clearly the change of time scale $\tau = t/\varepsilon$ reduces equation (1) to the standard form.

Also, a system of n weakly coupled harmonic oscillators

(2) $$\ddot{u}_k + w_k\, u_k = \varepsilon\, f_k(u_1,\ldots,u_k,\dot{u}_1,\ldots,\dot{u}_k)$$
$$k = 1,2,\ldots,n$$

can be written in the standard form. In fact, using the amplitude and phase
variables $y = (y_1,\ldots,y_n)$ and $\theta = (\theta_1,\ldots,\theta_n)$ defined by

$$\dot{u}_k + i\, w_k \dot{u}_k = y_k\, e^{i(\theta_k + w_k t)}$$

the system (2) can be transformed into

$$\dot{\theta} = \epsilon\, F(t,\theta,y)$$
$$\dot{y} = \epsilon\, G(t,\theta,y).$$

2. The *Averaging Method*, notably developed by Krylov, Bogolyubov and Mitro-
polsky [1], consists of approximating the solutions of the equation (E) by the
solution of the autonomous equation

(AE) $$\dot{\bar{x}} = \epsilon\, f_0(\bar{x})$$

where $f_0(\bar{x})$ is the mean value of $f(t,\bar{x},\epsilon)$ with respect to t at $\epsilon = 0$.
That is to say,

$$f_0(\bar{x}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t,\bar{x},0)dt.$$

Equation (AE) is called the *averaged equation* and it simplifies equation (E)
by at least reducing the dimension of the system by one.

To see that this is a reasonable procedure one rescales time $(\tau = \epsilon t)$ to
obtain the equations

(E') $$\frac{dx}{d\tau} = f(\tau/\epsilon,x,\epsilon)$$

(AE') $$\frac{d\bar{x}}{d\tau} = f_0(\bar{x}).$$

Thus, we have that for small values of ϵ, the equation (E') has a very rapidly
oscillating forcing term and we would expect that in the first approximation
the solution, $x_\epsilon(\tau)$, of (E) will only obey the average effect of the forcing.
For the case when the right hand side of equation (E) is periodic with respect
to t, the validity of this reasoning is justified by the fact that, the dif-
ference between the integral of a periodic function and its mean value tends to
zero as the period tends to zero.

3. The following *decomposition theorem* (Hale [2]) is a fundamental result of
the averaging theory. It says that after a change of variables the equation (E)
is just a perturbation of the averaged equation. In view of this result the
averaging procedure can be treated, in many respects, as a classical regular
perturbation method.

THEOREM 1. <u>Given any compact set</u> $\Omega \subset \mathbb{R}^n$ <u>there is an</u> $\varepsilon_0 > 0$ <u>and a</u> <u>function</u> $u(t,y,\varepsilon)$ <u>continuous on</u> $\mathbb{R} \times \mathbb{R}^n \times (0,\varepsilon_0]$ <u>such that</u>:

i) $u(t,y,\varepsilon)$ <u>is almost periodic in</u> t <u>uniformly with respect to</u> y <u>in compact sets for each fixed</u> ε;

ii) $u(t,y,\varepsilon)$ <u>has a continuous derivative with respect to</u> t <u>and derivatives of any arbitrary specified order with respect to</u> y;

iii) $\varepsilon u(t,y,\varepsilon)$, $\varepsilon \frac{\partial u}{\partial y}(t,y,\varepsilon)$ <u>tend to zero as</u> $\varepsilon \to 0$ <u>uniformly with respect to</u> $t \in \mathbb{R}$ <u>and</u> y <u>in compact sets</u>;

iv) <u>The change of variables</u>

(3)
$$x = \begin{cases} y + \varepsilon u(t,y,\varepsilon) & \text{for } (t,y,\varepsilon) \in \mathbb{R} \times \Omega \times (0,\varepsilon_0] \\ y & \text{for } (t,y,\varepsilon) \in \mathbb{R} \times \Omega \times \{0\} \end{cases}$$

<u>transforms equation</u> (E) <u>into</u>

$$\dot{y} = \varepsilon f_0(y) + \varepsilon g(t,y,\varepsilon)$$

This function, $g(t,y,\varepsilon)$, is continuous, has continuous derivative with respect to y on $\mathbb{R} \times \Omega \times [0,\varepsilon_0)$ and approaches zero as $\varepsilon \to 0$ uniformly with respect to $t \in \mathbb{R}$ and x in compact sets.

For the case in which the right hand side of the equation (E) is periodic in t, it is an easy exercise to prove that the transformation (3) with

(4)
$$u(t,y,\varepsilon) = \int_0^t [f(s,y,0) - f_0(y)]ds$$

produces the desired change in equation (E).

The case when $f(t,x,\varepsilon)$ is almost periodic (a.p) is not as simple because then, the function u defined by the equation (4) is not necessarily a.p. with respect to t. It happens that the integral of an a.p. function with mean value zero is not always a.p. and in fact can be unbounded [3]. However, it can be proved that there exists a function $u(t,x,\varepsilon)$ a.p. in t uniformly with respect to x in compact sets such that

$$\frac{\partial u}{\partial t}(t,y,\varepsilon) \to f(t,y,0) - f_0(y)$$

uniformly with respect to $t \in \mathbb{R}$ and y in compact sets. Using this function u in the transformation (3), the proof of Theorem 1 follows in essentially the same way as in the periodic case.

4. In the following sections we will pay attention to the problem of the validity of the averaging method. This is the problem of justifying and giving conditions under which we can obtain information about the solutions of the full equation (E) by studying instead the averaged equation (AE). This question has been approached in different ways; there is, for instance, a class of results that guarantee the existence of almost periodic solutions of the full equation

under the assumption that the averaged equation has a stable equilibrium that satisfies a non-degeneracy condition. Here we will be interested in finding conditions under which two solutions of equations (E) and (AE) with the same initial condition approach each other as ε tends to zero, uniformly in t. We will be also interested in comparing solutions that start with nearby initial conditions.

Section 5 is devoted to the uniform validity of the approximations during finite intervals of time and the sections 6 and 7 to the validity over infinite intervals of the form $[t_0, \infty)$. The results that will be presented in Section 7 are going to be particularly useful to conclude that the averaged equation reflects the qualitative behavior of the full system.

5. It follows from Theorem 1 that the equation (E') can be transformed into

(5) $$\frac{dy}{d\tau} = f_0(y) + g(\tau, y, \varepsilon)$$

with the function g continuous and $g(\tau, y, \varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly for $t \in \mathbb{R}$ and x in compact sets. If $\phi(\tau)$ is a solution of (AE') defined for all positive τ, then by the continuity of the solutions of equation (5) with respect to changes in the initial conditions and the parameter ε, we have that for a given η, $T > 0$ there are positive numbers ε_0 and δ such that $0 < \varepsilon < \varepsilon_0$ and $|y_0 - \phi(0)| < \delta$ implies that the solution of (5), $y_\varepsilon(\tau)$, with initial condition $y_\varepsilon(0) = y_0$, satisfies $|y_\varepsilon(\tau) - \phi(\tau)| < \eta$ for any $\tau \in [0,T]$. It can also be proved that because of the nature of the change of variables (3), the same is true when $y_\varepsilon(\tau)$ is a solution of (AE'). From this and taking in account that between the system (E) and the system (E') the rescaling $\tau = \varepsilon t$ is involved, we obtain the following result.

THEOREM 2. For each positive value of ε, let $\phi_\varepsilon(t)$ be a solution of the equation (AE). Assume that for some $\varepsilon > 0$, ϕ_ε is defined for all $t \in [0,\infty)$. Then, given a tolerance $\eta > 0$ and T as big as we please, there are positive numbers $\varepsilon_0(\eta,T)$ and $\delta(\eta,T)$ such that $0 < \varepsilon < \varepsilon_0$ and $|x_\varepsilon(0) - \phi_\varepsilon(0)| < \delta$ implies that

$$|x_\varepsilon(t) - \phi_\varepsilon(t)| < \eta \quad \text{for all } t \in [0,T/\varepsilon].$$

Here x_ε represents a solution of equation (E).

This is a more general version of the Bogolyubov's theorem [4] on the validity of the averaging approximations over the so called long intervals of time. The further generality here is in the fact that we are not restricted to compare solutions of the equations (E) and (AE) that start with the same initial conditions but neighboring initial conditions are allowed.

6. The Theorem 2 justifies the application of the averaging procedure during intervals of time as long as we want by taking ε small enough. Nevertheless it has the following weakness: it could be that no matter how small we take the value of ε, there is a moment after which the approximation fails. This situation would be problematic if we are interested in the long run behavior of a dynamical system. Figure 1 illustrates this phenomenon.
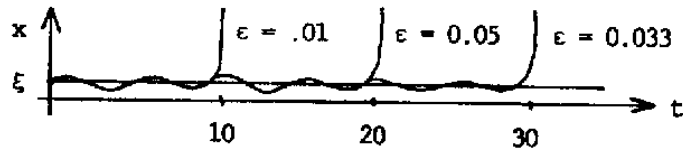

Figure 1. In the figure it is supposed that $x = \xi$ is a solution of the averaged equation and the other curves are solutions of the exact equation for different values of ε.

The system

(E)

$$\dot{x}_1 = \varepsilon x_2$$

$$\dot{x}_2 = -\varepsilon x_1 + \varepsilon \operatorname{sen} \varepsilon t$$

shows us that even for solutions that are Lyapunov stable, the situation illustrated in Figure 1 can actually happen. This is a linear system with resonant forcing and has the orbit structure depicted in Figure 2(a). Averaging the right hand side we are left with the equation

(AE)

$$\dot{\bar{x}}_1 = \varepsilon \bar{x}_2$$

$$\dot{\bar{x}}_2 = -\varepsilon \bar{x}_1$$

and this has a phase portrait like the one of Figure 2(b). Clearly, the periodic solutions of (AE) can not approximate the unbounded solutions of the equation (E) uniformly for $t \in [0,\infty)$.

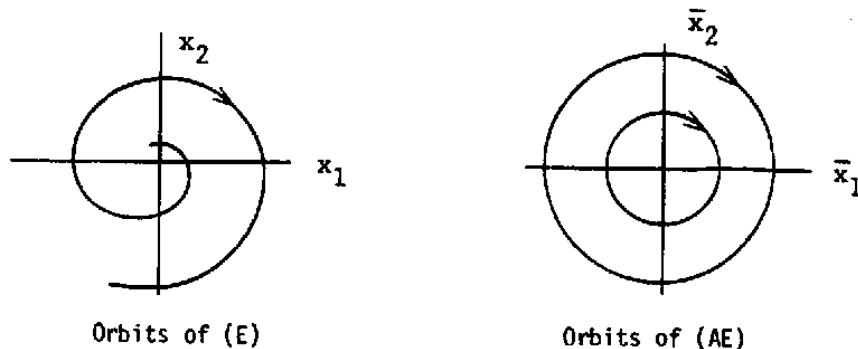Orbits of (E)   Orbits of (AE)

Figure 2

Once it has been established that the averaging approximation can break down during the scale of time $[t_0, \infty)$, it remains to find out under what conditions it is valid. It has been proved by Volosov [5],[6], Banfi [7] and Eckhaus [8] that for solutions with strong stability properties the approximation is justified over the unbounded interval $[t_0, \infty)$. The results that will be presented in the next section, extend and complement these previous results by treating, in a different way, more general solutions and giving us global information.

7. Let be $\phi: \mathbf{R} \times (0, \infty) \to \mathbf{R}^n$ and let us denote by x_ε a solution of equation (E). We will say that $\phi(t, \varepsilon)$ has a *stable neighborhood* (*s.n.*) as $\varepsilon \to 0$ for the equation (E), if for any $\eta > 0$ there are positive numbers $\varepsilon_0(\eta)$ and $\delta(\eta)$ such that $(t_0, \varepsilon) \in [0, \infty) \times (0, \varepsilon_0)$ and $|x_\varepsilon(t_0) - \phi(t_0, \varepsilon)| < \delta$ implies that $|x_\varepsilon(t) - \phi(t, \varepsilon)| < \eta$ for all $t \geq t_0$.

Similarly if for each $\varepsilon > 0$, γ_ε is a set in $\mathbf{R}^n$, we will say that γ_ε has an *orbitally stable neighborhood* (*o.s.n.*) as $\varepsilon \to 0$ for the equation (E) if for any $\eta > 0$ there are positive numbers $\varepsilon_0(\eta)$ and $\delta(\eta)$ such that $(t_0, \varepsilon) \in [0, \infty) \times (0, \varepsilon_0)$ and $d(x_\varepsilon(t_0), \gamma) < \delta$ implies that $d(x_\varepsilon(t), \gamma) < \eta$ for all $t \geq t_0$. Here $d(x, \gamma)$ represents the distance between the point x and the set γ.

THEOREM 3. (Ref. [9]). Let ϕ be a solution of $\dot{x} = f_0(x)$ and $x_\varepsilon(t; t_0, x_0)$ the solution of equation (E) that satisfies the initial condition $x_\varepsilon(t_0; t_0, x_0) = x_0$.

(a) If ϕ is uniformly asymptotically stable and bounded then the solution of the averaged equation, $\bar{x}_\varepsilon(t) = \phi(\varepsilon t)$, has a *s.n.* as $\varepsilon \to 0$ for equation (E). Also, if x_0 is a point in the domain of attraction of ϕ, then given $\eta > 0$, there is $\varepsilon_0(\eta) > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there exists $T(\varepsilon) > 0$ such that $|x_\varepsilon(t; t_0, x_0) - \phi(\varepsilon t)| < \eta$ for all $t \geq T$.

(b) If ϕ is orbitally uniformly asymptotically stable and bounded with orbit γ then γ has an *o.s.n.* as $\varepsilon \to 0$ for the equation (E). If x_0 is a point in the domain of attraction of γ, then given $\eta > 0$ there is $\varepsilon_0(\eta) > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there exists $T(\varepsilon) > 0$ such that $d(x_\varepsilon(t; t_0, x_0), \gamma) < \eta$ for all $t \geq T$.

Remarks. (i) The hypothesis of (a) and (b) can be satisfied for solutions that are not linearly stable. Thus, the assertions of the theorem can be also true for solutions that do not attract exponentially.

(ii) The assertion of (a) also holds when ϕ is an asymptotically stable static solution (see appendix). The same happens for (b) when ϕ is an orbitally asymptotically stable periodic solution (stable limit cycle).

iii) The statement (a) is also true for any solution stable under persistent disturbances and bounded. Furthermore, if we extend in a natural way the notion

of stability when persistent disturbances for sets in R^n then we can obtain a similar result to that of (b) for this more general situation [10].

EXAMPLE 1. The equation

(6)
$$\dot{x} = \epsilon y(\cos^2 t - z)$$
$$\dot{y} = -\epsilon x(1-2z + xy)$$
$$\dot{z} = -\epsilon z + \epsilon \, \text{sen}(t + xy)$$

has not $(x,y,z) = (0,0,0)$ as a solution. However, $(0,0,0)$ is a solution of the averaged system

(7)
$$\dot{x} = \epsilon y(\tfrac{1}{2} - z)$$
$$\dot{y} = -\epsilon x(1 - 2z + xy)$$
$$\dot{z} = -\epsilon z.$$

Since $V = x^2 + \frac{1}{2} y^2 + z^2$ is a Lyapunov function for this system (there are not invariant sets in the x and y axis other than the origin) we have that $(0,0,0)$ is asymptotically stable with some domain of attraction D. Then by Theorem 3, the origin has a stable neighborhood for equation (6), and for small ϵ all solutions of (6) that start on D will end up in a small neighborhood of the origin.

EXAMPLE 2. The planar system

(8)
$$\dot{x} = -\epsilon y + z\epsilon x \, \text{sen}^2 t(1 - \sqrt{x^2+y^2})^3$$
$$\dot{y} = \epsilon x + \epsilon y(1 - \sqrt{x^2+y^2})^3 + \epsilon \cos(t + x)$$

has, after averaging, the polar coordinates expression

(9)
$$\dot{\gamma} = \epsilon \gamma(1 - \gamma)^3$$
$$\dot{\theta} = \epsilon.$$

The unit circle is a stable limit cycle for this equation. Then by Theorem 3(b) it has an $o.s.n.$ for equation (8). We also conclude that for small ϵ all solutions of (8) except the origin will be eventually within a small neighborhood of the unit circle.

APPENDIX. Let $x(t;t_0,x_0)$ denote the solution of the equation

(10) $\dot{x} = f(t,x)$

that satisfies the initial condition $x(t_0;t_0,x_0) = x_0$. A solution ϕ of (10) is called *uniformly stable* if it is defined for $t \geq 0$ and given any $\eta > 0$ there is $\delta(\eta)$ such that for any $t_0 \geq 0$, $|x_0 - \phi(t_0)| < \delta$ implies that $|x(t;t_0,x_0) - \phi(t)| < \eta$ for all $t \geq t_0$. ϕ is called *uniformly asymptotically stable* if it is uniformly stable and there exists $h > 0$ with the property

that for every $\eta > 0$ there is $T(\eta) > 0$ such that for any $t_0 \geq 0$, $|x_0 - \phi(t_0)| < h$ implies that $|x(t;t_0,x_0) - \phi(t)| < \eta$ for $t \geq t_0 + T$. Observe that h is independent of t_0 and T is independent of both t_0 and x_0. Let us call γ the positive semiorbit of ϕ in the phase space, that is $\gamma = \{x \in \mathbb{R}^n \mid x = \phi(t)$ for some $t \geq 0\}$. We say that ϕ is *orbitally uniformly stable* if for any $\eta > 0$ there is $\delta(\eta)$ such that for any $t_0 \geq 0$, $d(x_0,\gamma) < \delta$ implies that $d(x(t;t_0,x_0),\gamma) < \eta$ for all $t \geq t_0$. Here $d(x,X)$ represents the distance between the point $x \in \mathbb{R}^n$ and the set $X \subset \mathbb{R}^n$. If there exists also $h > 0$ with the property that for every $\eta > 0$ there is $T(\eta) > 0$ such that for any $t_0 \geq 0$, $d(x_0,\gamma) < h$ implies that $d(x(t;t_0,x_0),\gamma) < \eta$ for $t \geq t_0 + T$, then we say that ϕ is *orbitally uniformly asymptotically stable*.

If f in equation (10) is periodic in t uniformly with respect to x, it is the case that for periodic solutions of the same period as f, simple stability in the sense of Lyapunov implies uniform stability, asymptotic stability implies uniform asymptotic stability. Orbital stability implies orbital uniform stability and orbital asymptotic stability implies orbital uniform asymptotic stability [11],[12].

BIBLIOGRAPHY

1. N. N. Bogolyubov and Y. A. Mitropolsky, Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordon and Breach Science Publishers, New York, 1961.

2. J. K. Hale, Ordinary Differential Equations (section v.3), Wiley Interscience, New York, 1969.

3. See appendix on almost periodic functions in reference 2.

4. See chapter on the foundations of the averaging method in reference 1.

5. V. M. Volosov, "Averaging in systems of differential equations," Russ. Math. Surveys, Vol. 17, No. 6, 1962.

6. V. M. Volosov, "Averaging on an unbounded interval," Soviet Mathematics, Translation of Doklady, Vol. III, Part II, 1962.

7. C. Banfi, "Sull'aprossimazione di processi non stationari in meccanica non lineari," Boll. Un. Mat. Italiana 22, 442-450.

8. V. Eckhaus, "New approach to the asymptotic theory of nonlinear oscillations and wave propogation," J. Math. Anal. Appl. 49, 575-611.

9. H. Carrillo Calvet, El Método de el promedio y estabilidad bajo disturbios persistentes con aplicación a sincronización racional (Doctoral Thesis), Facultad de Ciencias, UNAM, 1984.

10. H. Carrillo Calvet, "Stability under persistent disturbances and averaging," in preparation.

11. J. L. Massera, "Contributions to stability theory," Ann. of Math. 64, 1956, 182-206.

12. T. Yoshizawa, "Stability of sets and perturbed systems," Funkcial. Ekvac., 5, 1963, 31-39.

Departamento de Matematicas
Facultad de Ciencias
Universidad Nacional Autonoma de México
D.F. 04510 MEXICO

Current Address:
Lefschetz Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, RI   02912