AVERAGING AND SYNCHRONIZATION OF WEAKLY COUPLED SYSTEMS

Humberto Carrillo Calvet[1]
Facultad de Ciencias
Universidad Nacional Autónoma de México,
D.F. 04510, México

Abstract

The averaging method for ordinary differential equations is briefly reviewed and new results on the uniform validity of its approximations over infinite intervals of time are given. As an application a synchronization theorem for weakly coupled systems is proved.

1. The Averaging Method

In this perturbation method one considers the equation

$$(E) \qquad \dot{x} = \varepsilon f(t,x,\varepsilon)$$

where $f\colon \mathbb{R} \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ is continuous, has a continuous partial derivative with respect to x and is almost periodic in t, uniformly with respect to (x,ε) in compact sets. Under these conditions $f(t,x,\varepsilon)$ has a mean value:

$$f_0(x) := \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t,x,0)\, dt.$$

The averaging method consists in approximating the solutions of equation (E) by the solutions of the *averaged equation*,

$$(AE) \qquad \dot{\bar{x}} = \varepsilon f_0(\bar{x}),$$

for small values of the parameter ε.

This procedure simplifies the original system (E) in, at least, reducing its dimension by one. The fact that it is widely used in nonlinear oscillation theory is going to be corroborated by its frequent appearance in the present collection of papers.

To give an intuitive justification of the method we will use a

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new time scale ($\tau = \varepsilon t$) in equation (E) and (AE) to have:

(E') $$\frac{dx}{d\tau} = f(\tau/\varepsilon,\ x,\ \varepsilon)$$

(AE') $$\frac{d\bar{x}}{d\tau} = f_0(\bar{x})$$

Now it is obvious that equation (E') involves two time scales: y varies according to the time scale in which τ is measured, but the system is being forced on a faster scale of time τ/ε. For small values of ε, $f(\tau/\varepsilon, x, \varepsilon)$ oscillates rapidly as τ changes. Therefore, it is reasonable to consider that in each moment τ, the solution y will feel only the average effect of the vector field $f(\tau/\varepsilon, x, \varepsilon)$ at the point $x(\tau)$, which is given by the mean value $f_0(x\ (\tau))$.

2. Foundations of the Averaging Method

The following result is a generalization of Bogolyubov's theorem on the validity of the averaging method over finite time intervals of the form $[t_0, \frac{1}{\varepsilon}]$.

Theorem 1. (Ref. [1], [2])

Let us denote by $x_\varepsilon(t)$ a solution of equation (E) for the value ε of the parameter. Assume that $\phi(t)$ is a solution of the equation $\dot{x} = f_0(x)$. (Under these conditions, for each $\varepsilon > 0$, the function $\bar{x}_\varepsilon(t) = \phi(\varepsilon t)$ is a solution of the averaged equation (AE)). Then, given a tolerance $\eta > 0$ and T as large as we please, there are ε_0 and δ, positive, such that $(t_0, \varepsilon)\ [0,\infty)x(0,\varepsilon_0)$ and $|x_\varepsilon(t_0) - \phi(\varepsilon t_0)| < \delta$ implies that

$$|x_\varepsilon(t) - \phi(\varepsilon t)| < \eta \qquad \text{for all } t\in[t_0, T/\varepsilon].$$

This theorem justifies the averaging approximation during intervals of time that can be as long as we wish, with the condition that ε is taken small enough. However, it does not follow that for two solutions $x_\varepsilon(t)$ and $\bar{x}_\varepsilon(t)$ that satisfy the same initial condition we will have that

$$\lim_{\varepsilon\to 0} |x_\varepsilon(t) - \bar{x}_\varepsilon(t)| = 0$$

uniformly with respect to t $(0,\infty)$. In fact, this is not true and it can happen that, without mattering how small we choose the value of ε, in the long run, the two solutions x_ε and $\bar{x}_\varepsilon$ find themselves very far apart from each other [2]. This constitutes a strong limitation of the method. However, it has been proved ([4], [5], [6], [7]) that the averaging approximation does not fail for strongly stable solutions. After giving some definitions, we will state a very general result on the uniform validity of the averaging method over infinite intervals of time.

Let be ϕ: $\mathbb{R} \times (0,\infty) \to \mathbb{R}^n$ and let us denote by x_ε a solution of equation (E). We will say that $\phi(t,\varepsilon)$ has a *stable neighborhood* (*s.n.*) as $\varepsilon \to 0$ for the equation (E), if for any $\eta > 0$ there are positive numbers $\varepsilon_0(\eta)$ and $\delta(\eta)$ such that $(t_0,\varepsilon) \in [0,\infty) \times (0,\varepsilon_0)$ and $|x_\varepsilon(t_0) - \phi(t_0,\varepsilon)| < \delta$ implies that $|x_\varepsilon(t) - \phi(t,\varepsilon)| < \eta$ for all $t > t_0$.

Similarly if for each $\varepsilon > 0$, γ_ε is a set in $\mathbb{R}^n$, we will say that γ_ε has an *orbitally stable neighborhood* (*o.s.n.*) as $\varepsilon \to 0$ for the equation (E) if for any $\eta > 0$ there are positive numbers $\varepsilon_0(\eta)$ and $\delta(\eta)$ such that $(t_0,\varepsilon) \in [0,\infty) \times (0,\varepsilon_0)$ and $d(x_\varepsilon(t_0),\gamma) < \delta$ implies that $d(x_\varepsilon(t),\gamma) < \eta$ for all $t \geq t_0$. Here $d(x,\gamma)$ represents the distance between the point x and the set γ.

Theorem 2. (Ref. [3]).

Let ϕ be a solution of $\dot{x} = f_0(x)$ and $x_\varepsilon(t;t_0,x_0)$ the solution of equation (E) that satisfies the initial condition $x_\varepsilon(t_0;t_0,x_0) = x_0$.

(a) If ϕ is uniformly asymptotically stable and bounded then the solution of the averaged equation, $\bar{x}_\varepsilon(t) = \phi(\varepsilon t)$, has a *s.n.* as $\varepsilon \to 0$ for equation (E). Also, if x_0 is a point in the domain of attraction of ϕ, then given $\eta > 0$ there is $\varepsilon_0(\eta) > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there exists $T(\varepsilon) > 0$ such that $|x_\varepsilon(t;t_0,x_0) - \phi(\varepsilon t)| < \eta$ for all $t \geq T$.

(b) If ϕ is orbitally uniformly asymptotically stable and bounded with orbit γ then γ has an *o.s.n.* as $\varepsilon \to 0$ for the equation (E). If x_0 is a point in the domain of attraction of γ, then given $\eta > 0$ there is $\varepsilon_0(\eta) > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there exists $T(\varepsilon) > 0$ such that $d(x_\varepsilon(t;t_0,x_0), \gamma) < \eta$ for all $t \geq T$.

Remarks.

(i) Neither (a) nor (b) require exponential stability of the solution ϕ, and then the theorem also applies in the nonlinear stability case.

(ii) In an autonomous equation asymptotically stable (*a.s.*) static solutions are uniformly *a.s.* and orbitally *a.s.* periodic solutions are orbitally uniformly *a.s.*. Then, these conditions are enough to guarantee the assertions (a) and (b) respectively.

(iii) The statement (a) is also true for any solution stable under persistent disturbances and bounded. Furthermore, if we extend in a natural way the notion of stability under persistent disturbances for sets in $\mathbb{R}^n$ then we can obtain a similar result to that of (b) for this more general situation [3].

Example. Consider the van der Pol equation with rapidly oscillating, forcing and coefficients.

$$\ddot{x} + \mu(t/\varepsilon)(x^2-1)\dot{x} + \omega(t/\varepsilon)\, x = g(t/\varepsilon,\, x,\, \dot{x}). \tag{1}$$

Let us assume that the functions μ and ω are almost periodic with positive mean values $\bar{\mu}$ and $\bar{\omega}$ respectively. Let us also assume that $g(t,x,y)$ is almost periodic in t uniformly with respect to (x,y) in compact sets, and has mean value zero. Reescaling the time ($\tau = t/\varepsilon$) and writing equation (1) as a system, we have

$$\frac{dx}{d} = \varepsilon y$$
(2)
$$\frac{dy}{d\tau} = -\varepsilon\mu(\tau)(x^2-1)\, y - \varepsilon\omega(\tau)x + g(\tau,x,y).$$

The corresponding averaged system

$$\frac{d\bar{x}}{d\tau} = \varepsilon\bar{y}$$
(3)
$$\frac{d\bar{y}}{d\tau} = -\varepsilon\bar{\mu}(\bar{x}^2-1)\bar{y} - \varepsilon\bar{\omega}\,\bar{x}$$

has a limit cycle. By Theorem 2 the set γ has an orbitally stable neighborhood, this means that the solutions of (2) that start close to γ remain close for all future time. Also, in its domain of attraction, γ is "attracting" the solutions of (E) in the sense of assertion (b).

The limit cycle of van der Pol equation is an exponential attractor. This is because its Poincaré map is linearly asymptotically stable. However, the theorem can also be applied to limit cycles that do not satisfy this property [2].

3. Synchronization.

Let us consider the equation

$$\dot{\theta} = f(t,\theta) \tag{4}$$

where $t \in \mathbb{R}$, $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ and f is 2π-periodic in each component of the vector θ. Phase-only equations like (4) appear in models of n coupled ring devices [8].

Let be $\omega = (\omega_1, \ldots, \omega_n)$ with each ω_i a natural number. We say that the system (4) has the *rational synchronization* property with *rotation vector* ω, if for each t_0 $\mathbb{R}$ there is an open set $\Gamma \subset \mathbb{R}^n$ such that, any solution $\theta(t)$ of (4) with $\theta(t_0) \in \Gamma$ satisfy the relations:

(5) $$\lim_{t \to \infty} \theta_1 : \theta_2 : \ldots : \theta_n = \omega_1 : \omega_2 : \ldots : \omega_n$$

That is,

$$\lim_{t \to \infty} \frac{\theta_i}{\theta_j} = \frac{\omega_i}{\omega_j} \quad \text{for } i, j = 1, \ldots, n.$$

Considering that equation (1) represents a one parameter family of vector fields on an n-dimensional torus, the condition (5) means that the orbit of $\theta(t)$ winds, asymptotically, ω_i times around the i-axis of this torus for each ω_j windings around the j-axis. To have rational synchronization with rotation vector $\omega = (1, \ldots, 1)$ means that the frequencies are asymptotically the same and it is called just *synchronization*.

4. Weakly coupled systems.

Here we will study a system in amplitude (x) and phase (θ) variables of the form:

(6)
$$\dot{x} = \varepsilon G(x, \theta, \varepsilon)$$
$$\dot{\theta} = \omega + \varepsilon F(x, \theta, \varepsilon)$$

where G: $\mathbb{R}^m \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^m$ and F: $\mathbb{R}^m \times \mathbb{R}^n \times [0, \infty)$ $\mathbb{R}^n$ are continuos, 2π-periodic in each component of the vector θ, of class C^1 with respect to (x, θ) for each ε fixed an F is bounded as a function of x for each (θ, ε) fixed.

Let $\mathbb{Z}$ denote the set of integer numbers. We will suppose that $\{\omega, \Omega_2, \ldots, \Omega_n\} \subset \mathbb{Z}^n$ is an orthogonal basis of $\mathbb{R}^n$ and A is the nxn matrix whose rows are $\omega, \Omega_2, \ldots, \Omega_n$. We also adopt the following notation:

$$G_0(x, u) = \frac{1}{\omega \cdot \omega} \lim_{T \to \infty} \frac{1}{T} \int_0^T G(x, A^{-1}\binom{v}{u}, 0) dv$$

$$F_0(x, u) = \frac{1}{\omega \cdot \omega} (F_0^2(x, u), \ldots, F_0^n(x, u))$$

where $u \in \mathbb{R}^{n-1}$, $v \in \mathbb{R}$, the dot is the scalar product of $\mathbb{R}^n$, and

$$F_0^i(x,u) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \Omega_i \cdot F(x, A^{-1}\binom{v}{u}, 0)\,dv$$

for $i = 2,\dots,n$.

The following theorem gives conditions under which the angular variables of the system (6) synchronizes rationally to ω for small values of the parameter ε; it generalizes a result by Hoppensteadt and Kenner (Ref. [9]).

<u>Theorem 3</u>.

If the auxiliary system

$$(7)\qquad \frac{dx}{dv} = G_0(x,u) \qquad \frac{du}{dv} = F_0(x,u)$$

has a bounded and orbitally uniformly asymptotically stable solution, then there exists and $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, the phase variables $(\theta_1,\dots,\theta_n)$ of the system (6) sinchronize rationally with rotation vector ω.

The proof of this theorem will give us an example of an argument that requires the validity of the averaging method over the unbounded interval of time $[t_0,\infty)$.

Proof. In the new variables $v = \omega \cdot \theta$ and $u_i = \Omega_i \cdot \theta$ for $i=2,\dots,n$, equation (6) becomes:

$$(8)\qquad \begin{aligned} \dot{x} &= \varepsilon G(x, A^{-1}\tbinom{v}{u}, \varepsilon) \\ \dot{u}_i &= \varepsilon \Omega_i \cdot F(x, A^{-1}\tbinom{v}{u}, \varepsilon) \qquad (\text{for } i = 2,\dots,n) \\ \dot{v} &= \omega\cdot\omega + \varepsilon\omega\cdot F(x, A^{-1}\tbinom{v}{u}, \varepsilon). \end{aligned}$$

Since $F(x, A^{-1}\binom{v}{u}, \varepsilon)$ is bounded, there exists $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon_1$ the component $v(t)$ of any solution of equations (8) tends monotonically to infinite.

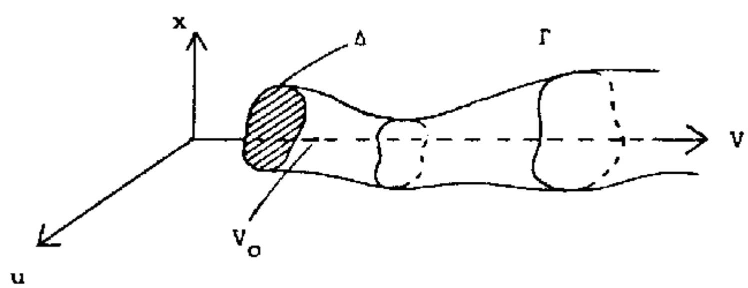
Eliminating time in equations (8) we obtain

$$\frac{dx}{dv} = \frac{\varepsilon G(x, A^{-1}\binom{v}{u}, \varepsilon)}{\omega\cdot\omega + \varepsilon\omega\cdot F(x, A^{-1}\binom{v}{u}, \varepsilon)}$$

(9)

$$\frac{du_i}{dv} = \frac{\varepsilon\Omega_i \cdot F(x, A^{-1}\binom{v}{u}, \varepsilon)}{\omega\cdot\omega + \varepsilon\omega\cdot F(x, A^{-1}\binom{v}{u}, \varepsilon)} \qquad (\text{for } 1 = 2, \ldots, n)$$

whose corresponding averaged equation is the auxiliary system (7). As this averaged system has an orbitally uniformly *a.s.* solution, Theorém 2 (b) implies the existence of $\Delta \subset \mathbb{R}^{m+n-1}$ and $\varepsilon_0 < \varepsilon_1$ such that, for $0 < \varepsilon < \varepsilon_0$, it happens that: if (x,u) is a solution of (9) and $(x(v_0), u(v_0)) \in \Delta$ with $v_0 > 0$, then (x,u) is bounded for $v > v_0$. Fix $v_0 > 0$ and let Γ be the set of (y,z,v) in $\mathbb{R}^m \times \mathbb{R}^{n-1} \times \mathbb{R}$ such that $v > v_0$ and $(y,z) = (x(v), u(v))$ with (x,u) a solution of equation (9) that satisfies $(x(v_0), u(v_0)) \in \Delta$. This set, Γ, is open and for



$\varepsilon < \varepsilon_0$, if (x,θ) is a solution of (6) with $(x(t_0), \theta(t_0)) \in \Gamma$, then, for $k = 2, \ldots, n$, the function $u_k(t) = \Omega_K \cdot \theta(t)$ is bounded for $t > t_0$. Hence,

$$\lim_{t\to\infty} \frac{\Omega_K \cdot \theta(t)}{\omega \cdot \quad (t)} = 0$$

From this it follows that

$$\lim_{t\to\infty} \frac{\theta(t)}{\omega\cdot\theta(t)} = \lim_{t\to\infty} \left[\frac{\omega}{\omega\cdot\omega} + \sum_{k=2}^{n} \frac{\Omega_k\cdot\theta(t)}{\omega\cdot\theta(t)} \frac{\Omega_k}{\Omega_k\cdot\Omega_k}\right] = \frac{\omega}{\omega\cdot\omega}$$

Therefore, for $\varepsilon < \varepsilon_0$, if $(x(t_0), \theta(t_0)) \in \Gamma$, then

$$\lim_{t\to\infty} \frac{\theta_i(t)}{\theta_j(t)} = \lim_{t\to\infty} \frac{\theta_i(t)}{\omega\cdot\theta(t)} \frac{\omega\cdot\theta(t)}{\theta_j(t)} = \frac{\omega_i}{\omega_j} \qquad (\text{for } i,j = 1, \ldots, n).$$

This proves Theorem 3.

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