PERGAMON

Chaos, Solitons and Fractals 11 (2000) 2203–2213

CHAOS
SOLITONS & FRACTALS

www.elsevier.nl/locate/chaos

Generalized coherent states and nonlinear dynamics of a lattice quasispin model

M. Agüero [a,*], G. Frias [a], F. Ongay [b]

[a] Departamento de Fisica Nolineal, Facultad de Ciencias, Universidad Autonoma del Estado de Mexico, Instituto Literario 100, Toluca 5000, Edo. de Mex., Mexico

[b] Departamento de Matematica, Facultad de Ciencias, Universidad Autonoma del Estado de Mexico, Instituto Literario 100, Toluca 5000 Edo. de Mex., Mexico

Accepted 28 July 1999

Abstract

The nonlinear lattice equation of the φ^6 theory is studied by using the technique of generalized coherent states associated to a $SU(2)$ Lie group. We analyze the discrete nonlinear equation with weak interaction between sites. The existence of saddles and centers is shown. The qualitative parametric domains which contain kinks, bubbles and plane waves were obtained. The specific implications of saddles and centers to the parametric first- and second-order phase transitions are identified and analyzed. © 2000 Elsevier Science Ltd. All rights reserved.

PACS: 11.27.+d; 11.30.Na; 05.70.Jk

1. Introduction

The many-body quantum systems in an actual situation are so complicated that they usually need a certain reduction procedure from the quantum description to a classical one. The coherent states (CS) are a popular class of research techniques used to find good solutions to hard problems in studying many-body quantum systems. In fact, such a procedure consists in choosing trial functions (i.e. some basis) which can be used for averaging the quantum Hamiltonian. We should be very careful in performing the choosing of the trial functions. As common for doing this one can use a search strategy, originally proposed by Schrödinger, which was motivated by natural use of inherent symmetries of the system. It often happens that CSs are most suitable minimizing the uncertainty relation.

The coherent state approach was then developed by Glauber, Klauder and Sudarshan in important application of quantum optics [1–3]. The extension of these ideas to other phenomena has been treated extensively by several authors [4–7]. Perelomov [8] proposed a method for constructing generalized coherent states (GCS) for Lie groups. The GCS method permits to study quantum systems in the semiclassical version of the theory since the coherent state manifold can be interpreted as the canonical phase space of the system.

In spin or "quasispin" Hamiltonians studies it is natural to use GCS constructed on the spin operators of the group $SU(2)$. Such states for arbitrary values of the spin j are those corresponding to points of the coset spaces $SU(2j+1)/(SU(2j) \otimes U(1))$. As is well known, the GCS can be defined in a certain

* Corresponding author. Tel.: +52-72-965556; fax: +52-72-965554.
E-mail address: mag@coatepec.uaemex.mx (M. Agüero).

PII: S0960-0779(99)00140-X

association to an arbitrary Lie group as following. The set of vectors $|\psi_m\rangle = T(g_m)|\psi_0\rangle$ with $g_m \in G/P$ we call a system of GCS on the group G with a referent vector $|\psi_0\rangle$ and $T(g_m)$ being the irreducible unitary representation of the Lie group. Different vectors (states) will correspond to elements g_m that belong to the factor space $M = G/P$. It is evident that P is a subgroup of the group G and we denote it as the stationary group of the state $|\psi_0\rangle$. Hence, it is enough to take one element of each class to describe the set of different states. From the geometrical point of view, the group G is treated as fiber-bundle space with a base $M = G/P$ and layer P. Then the choosing of g_m corresponds to some section of this fiber-bundle space.

In this paper we will deal with the Lie group $G = SU(2)$. It is known that the system of GCS constructed on the $SU(2)/U(1)$ coset space may be written as

$$|\psi\rangle = T(g)|\psi_0\rangle = \mathrm{e}^{\alpha S^+ - \bar{\alpha} S^-}|0\rangle = \left(1+|\psi|^2\right)^{-j}\mathrm{e}^{\psi S^+}|0\rangle. \tag{1}$$

Here $\hat{S}^{\pm} = \hat{S}^x + \mathrm{i}\hat{S}^y$ and $\hat{S}^z$ are the generators for the algebra of the $SU(2)$ group, $\psi = (\alpha/|\alpha|)\tan|\alpha|, \alpha, \psi$ are complex numbers, $|0\rangle = |j, -j\rangle$ is the ground state and j defines the unitary representation of the group $SU(2)$. The set of trial functions (1) is seen to have the symmetry of sphere. For $j = 1$ the corresponding CS read [15]

$$|\psi\rangle = \frac{1}{1+|\psi|^2}\{|0\rangle + \sqrt{2}\psi|1\rangle + \psi^2|2\rangle\} \tag{2}$$

with $(|i\rangle, i = 0, 1, 2)$ being the pure spin states (down, middle and up states as usual). The components of the classical spin vector, $\vec{S} = (S^x, S^y, S^z) = \langle\psi|\hat{S}|\psi\rangle$ and of the quadrupole moment Q^{ij} for the GCS in the coset space $SU(2)/U(1)$ for other values of j are

$$S^+ = \bar{S}^- = 2j\frac{\bar{\psi}}{1+|\psi|^2}, \quad S^z = -j\frac{1-|\psi|^2}{1+|\psi|^2}, \quad Q^{zz} = \frac{j^2\left(1-|\psi|^2\right) + 2j|\psi|^2}{\left(1+|\psi|^2\right)^2}. \tag{3}$$

On the other hand, in many branches of theoretical physics the so-called φ^6-theory is a successful model to study peculiar properties of nonlinear waves, magnetization, nuclear hydrodynamics and so on (see for example [9–13]). The nonrelativistic version of this theory is the correct model to describe the envelope's propagation of the light pulse in dispersive potentials with either a saturable or higher' order refraction index [14]. The ϕ^6-model of complex scalar field also shows a rather rich phase transition picture, depending on the initial configuration and on the form of potential [17,18].

By focusing on dynamical behavior we proceed to study a 1D quantum quasispin ϕ^6-model obtained by Masperi et al. [19]. As it is well known a few 1D many-body problems can be exactly solved by utilizing a powerful Bethe ansatz for example. In this situation, it is difficult, due to absence of a systematic method, to obtain some exact results of 1D many-body systems. The system that we will study is not integrable. However, we will use the GCS to obtain appropriated results concerning to the dynamics of the model.

We wish to emphasize that in this article we are mainly concerned in using the method of GCS to get physical information specially by studying the singular points of the simplest dynamical system of this theory. Several results that in a great manner qualitatively converge with those previous done on this subject, mainly concerning the parametric first- and second-order phase transition have been obtained. We have captured in this simple fashion of a first approximation of the lattice equation, valuable information about the role of bubble and kink solitons in the parametric phase transition. Of course, many quantum features using this approach were disregarded, instead we get some averaged features that in some sense resemble the classical behavior.

The paper is organized as follows. In Section 2 we deal with the study of the lattice equations generated by using the GCS. Section 3 contains several results concerning the behavior of the singular points for two independent fields. The right implication of the obtained bifurcations to the parametric first- and second-order phase transition is analyzed in Section 4. Conclusions are posed in Section 5.

2. Lattice equations

The starting point is a lattice version of real field theory with self-interaction in $1+1$ dimension described by the quantum Hamiltonian obtained by Masperi and coworkers in [19]:

$$H=\sum_m\left\{\begin{pmatrix}0&0&0\\0&-\varepsilon&0\\0&0&-\kappa\end{pmatrix}_m-\delta\begin{pmatrix}0&1&0\\1&0&\alpha\\0&\alpha&0\end{pmatrix}_m\begin{pmatrix}0&1&0\\1&0&\alpha\\0&\alpha&0\end{pmatrix}_{m+1}\right\}. \tag{4}$$

where $\varepsilon,\kappa,\delta$ are the relevant parameter of the model, $\alpha=(\varepsilon/(\kappa-\varepsilon))^{1/2}$. This lattice version of the ϕ^6-model has a symmetry very similar to that of the spin model with spin value $s=1$ with the pure spin states up, middle and down. This property explicitly depends on the parameter α involved in the quantum Hamiltonian.

Let us analyze the quantum Hamiltonian (4) via the spin CS (2) constructed on $SU(2)/U(1)$. First, we have to evaluate the correlations for S^iS^j. It is simple to conclude that, since the spin operators at different lattice site commute, we have for all of them the relation.

$$\langle\psi|\hat{S}^i_m\hat{S}^k_{m+1}|\psi\rangle=\langle\psi|\hat{S}^i_m|\psi\rangle\langle\psi|\hat{S}^k_{m+1}|\psi\rangle,$$

where $|\psi\rangle=|\psi\rangle_m|\psi\rangle_{m+1}$.

In order to have the classical counterpart to the discrete quasi-spin model, we average its Hamiltonian (4), with the help of GCS (2) and get

$$H=\sum_m\frac{-\kappa-2\varepsilon|\psi_m|^2}{\left(1+|\psi_m|^2\right)^2}-2\delta\frac{\left(\psi_m+\bar{\psi}_m\right)\left(\psi_{m+1}+\bar{\psi}_{m+1}\right)\left(\alpha+|\psi_m|^2\right)\left(\alpha+|\psi_{m+1}|^2\right)}{\left(1+|\psi_m|^2\right)^2\left(1+|\psi_{m+1}|^2\right)^2}. \tag{5}$$

Let us now turn to the analysis of the classical lattice equation of motion for the system with Hamiltonian (5). As it was shown in [14–16] by applying the technique of GCS to a spin-one (or quasi spin-one, i.e. a system with three eigenstates) quantum system the classical equations that govern spin dynamics along with its quadrupole moment were obtained. In doing so the authors used $SU(2)/U(1)$ GCS to derive a classical Hamiltonian and the functional integral method to obtain the equations which read

$$\mathrm{i}\dot{\psi}_m=(1+|\psi_m|^2)^2\frac{\partial H}{\partial\overline{\psi_m}}, \tag{6}$$

where $H=\sum_m H_m$ is the classical lattice Hamiltonian and $H_k=\langle\psi_k|H|\psi_k\rangle$ at a lattice site. The Lagrangian (at a site) obtained takes the form

$$\mathscr{L}=\frac{\mathrm{i}\hbar}{1+|\psi|^2}\left(\psi\dot{\overline{\psi}}-\overline{\psi}\dot{\psi}\right)-\mathscr{H},$$

where $\mathscr{H}$ is the classical Hamiltonian at a lattice site.

These equations describe some averaged rather than complete dynamics for all the quantum fluctuations and some other quantum effects were disregarded. The Poisson bracket for two arbitrary functions A and B that both depend on $\overline{\psi}$ and ψ is

$$\{A,B\}_j=\mathrm{i}\frac{\left(1+|\psi|^2\right)^2}{2\hbar j}\left(\frac{\partial A}{\partial\overline{\psi}}\frac{\partial B}{\partial\psi}-\frac{\partial A}{\partial\psi}\frac{\partial B}{\partial\overline{\psi}}\right).$$

Let us now apply these results to the quasispin system. First of all we divide this problem into two parts. It can be easily observed from the Hamiltonian (4), that the lattice version of the model supports two interesting cases: $\alpha=1$, and $\alpha\neq1$. Let us consider these two cases

(a) *Case*: $\alpha=1$ or $2\varepsilon=\kappa$. After substituting the Hamiltonian (5) into Eq. (6), we can obtain the lattice equation of motion

$$-\frac{\mathrm{i}}{2}\dot{\psi}_m = -\varepsilon\psi_m + \delta(1-\psi_m^2)\left\{\frac{\overline{\psi_{m+1}}+\psi_{m+1}}{1+|\psi_{m+1}|^2}+\frac{\overline{\psi_{m-1}}+\psi_{m-1}}{1+|\psi_{m-1}|^2}\right\},$$

which in the small amplitude region is transformed to:

$$-\frac{\mathrm{i}}{2}\dot{\psi}_m = \left[-\varepsilon-\delta\psi_m\left(\psi_{m+1}+\psi_{m-1}+\text{c.c.}\right)\right]\psi_m+\delta\left[\left(\psi_{m+1}\right)\left(1-|\psi_{m+1}|^2\right)+\left(\psi_{m-1}\right)\left(1-|\psi_{m-1}|^2\right)+\text{c.c.}\right] \tag{7}$$

The quantum Hamiltonian (4) after substraction of a constant, becomes

$$\mathscr{H}_2 = -\sum_m \left(2\delta\, S_m^x S_{m+1}^x - \varepsilon S_m^z\right), \tag{8}$$

where S_m^z and S_m^x are the components of the spin operator $\hat{S}$ acting at site m. This expression by using relations (3) can also be written in the stereographic projection as

$$\mathscr{H}_2 = \sum_m \varepsilon\frac{1-|\psi_m|^2}{1+|\psi_m|^2} - 2\delta\left(\frac{\bar{\psi}_m+\psi_m}{1+|\psi_m|^2}\right)\left(\frac{\bar{\psi}_{m+1}+\psi_{m+1}}{1+|\psi_{m+1}|^2}\right). \tag{9}$$

This model was shortly treated in [16] and the solitonic treatment of this version of the model should be reported elsewhere.

(b) *Case*: $\alpha \neq 1$, Eq. (6) yields

$$\mathrm{i}\dot{\psi}_m = \frac{2\psi_m(\kappa-\varepsilon)+2\varepsilon\psi_m|\psi_m|^2}{\left(1+|\psi_m|^2\right)} - 2\delta\frac{\left(\alpha+(1-2\alpha)\psi_m^2+(2-\alpha)|\psi_m|^2-\psi_m^2|\psi_m|^2\right)}{\left(1+|\psi_m|^2\right)}$$

$$\times\left(\frac{\left(\psi_{m+1}+\text{c.c.}\right)\left(\alpha+|\psi_{m+1}|^2\right)}{\left(1+|\psi_{m+1}|^2\right)^2}+\frac{\left(\psi_{m-1}+\text{c.c.}\right)\left(\alpha+|\psi_{m-1}|^2\right)}{\left(1+|\psi_{m-1}|^2\right)^2}\right). \tag{10}$$

It is expected that this model will support a rich variety of ground states, for example, kink and bubble ground states. The complete study of the ground states of this theory is outside the present work. However, we could predict some representatives of possible ground states by making use of a suitable transformation of the quasispin Hamiltonian. By observing the unitary transformation that was done in [20], for the quasispin Hamiltonian (5) it is reasonable to suggest like in the ferromagnetic case, that the ground state will be close to the one for which [15].

$\psi_m = \psi_l$ with $l \neq m$.

If so, the values of the ground states are obtained by solving the following algebraic equation:

$$-2\Delta\left(\alpha+(1-2\alpha)\psi_m^2+(2-\alpha)|\psi_m|^2-\psi_m^2|\psi_m|^2\right)\times\frac{\left(\psi_m+\overline{\psi_m}\right)\left(\alpha+|\psi_m|^2\right)}{\left(1+|\psi_m|^2\right)^2}+\psi_m\left(1-\sigma+\sigma|\psi_m|^2\right)=0, \tag{11}$$

where we have two relevant positive parameters: $\sigma = \varepsilon/\kappa$ and $\Delta = \delta/\kappa$. One solution of the Eq. (11) is obvious $\psi_m = 0$. Other solutions can be obtained by representing $\psi_m = z\exp(\mathrm{i}\theta)$ and solving simultaneously equations for the real and imaginary part of ψ_m. The real values of the ground states are obtained only in the case when $\cos\,\theta = 1$, then

$$1-\sigma+\sigma z^2-\frac{4\Delta(\alpha+z^2)}{(1+z^2)^2}\left[\alpha+3(1-\alpha)z^2-z^4\right]=0. \tag{12}$$

This relation yields an algebraic cubic equation for z^2. As is known from algebra we have in this case various regions in the plane (σ, Δ) where the ground states acquire real values. For example, let us see what kind of dependence arises between the relevant parameters in a very particular case. Eq. (12) has two imaginary and one real root if the product of the coefficients of the terms z^4 and z^2 is equal to the free member. For these special ground states, the relation between the parameters (σ, Δ) is

$$\left[1-\sigma+12\Delta+16\Delta\sqrt{\frac{\sigma}{1-\sigma}}\right]\left[2-\sigma+12\frac{\sigma}{1-\sigma}-11\sqrt{\frac{\sigma}{1-\sigma}}\right]+\sigma+4\Delta\frac{\sigma}{1-\sigma}=0.$$

3. On certain properties of dynamics

The most natural way to predict the behavior of a chain seems to be making use of the "difference" equation (13) as the governing equation and simulating it numerically as it is correct in the sense of Hadamard. The problem is that too many computational points will be needed for direct numerical simulations if one is to model even the smallest system of relevance. As our system is not integrable, then unfortunately little theory is possible once this property is lost.

To overcome this difficulty we analyze Eq. (10) in the simple fashion of small amplitude version. This lattice equation can be cast in the form

$$\begin{aligned} \mathrm{i}\dot{\psi}_m &= 2(k-\varepsilon)\psi_m+2(2\varepsilon-k)\psi_m|\psi_m|^2-2\delta\left(\psi_{m+1}+\overline{\psi_{m+1}}+\psi_{m-1}+\overline{\psi_{m-1}}\right)\Big(\alpha^2+\alpha(1-2\alpha)\psi_m^2 \\ &\quad+\alpha(2-\alpha)|\psi_m|^2\Big)-2\delta\alpha(1-2\alpha)\left[|\psi_{m+1}|^2\left(\psi_{m+1}+\overline{\psi_{m+1}}\right)+|\psi_{m-1}|^2\left(\psi_{m-1}+\overline{\psi_{m-1}}\right)\right]+0(\psi^4). \end{aligned} \quad (13)$$

In the first approximation, for the sake of simplicity, let us consider the small interaction between sites and consequently neglect terms that are proportional to quadratic value of the intervicinity distances. By forgetting the mean of the field ψ_m, for a while, let us consider $\psi_m = \zeta$. Then, in the first approximation one can see that the equation of motion of the system acquires the form

$$\mathrm{i}\dot{\zeta}=2(k-\varepsilon)\zeta+2(2\varepsilon-k)\zeta|\zeta|^2-4\delta\left(\zeta+\overline{\zeta}\right)\left[\alpha^2+\alpha(1-2\alpha)\zeta^2+3\alpha(1-\alpha)|\zeta|^2\right].$$

By taking the standard form: $\zeta = x + \mathrm{i}y$, one obtains the system of equations

$$\begin{aligned} \dot{x} &= a_1y+a_2x^2y+a_3y^3 \\ \dot{y} &= b_1x+b_2y^2x+b_3x^3 \end{aligned} \quad (14)$$

with the following values of the constants:

$$a_1=2(1-\sigma),\quad a_2=2(2\sigma-1-8\Delta\alpha(1-2\alpha)),\quad a_3=2(2\sigma-1)$$
$$b_1=-2\left(1-\sigma-4\Delta\alpha^2\right),\quad b_2=-2(2\sigma-1-4\Delta\alpha(2-\alpha)),\quad b_3=-2(2\sigma-1-4\Delta\alpha(4-5\alpha)).$$

A further analysis of the above system of equation (14) is realized in the vicinity of the singular points (x_0, y_0) which satisfy the algebraic equations

$$\begin{aligned} x_0 &= \pm\sqrt{\frac{(2\sigma+\alpha\sigma-2)}{8\alpha\Delta(5\alpha-2\alpha^2-2)}}, \\ y_0 &= \pm\sqrt{\frac{(8\alpha^2\Delta-16\alpha^3\Delta-2\sigma-\sigma\alpha)}{8\alpha\Delta(5\alpha-2\alpha^2-2)}}, \end{aligned} \quad (15)$$

The general outlook of the linearized equation for Eq. (14) along singular points (x_0, y_0) is the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi\\ \eta\end{pmatrix}=\begin{pmatrix}2a_2x_0y_0 & 3a_3y_0^2+a_2x_0^2+a_1\\ 3b_3x_0^2+b_2y_0^2+b_1 & 2b_2x_0y_0\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}.$$

Here we considered the linear perturbation to the variables: $x = x_0 + \xi$, $y = y_0 + \eta$. It is very easy to verify that the eigenvalues are obtained resolving this equation:

$$\lambda^2 - 2x_0y_0(a_2 + b_2)\lambda + \left(4a_2b_2x_0^2y_0^2 - \left(3a_3y_0^2 + a_2x_0^2 + a_1\right)\left(3b_3x_0^2 + b_2y_0^2 + b_1\right)\right) = 0. \tag{16}$$

Let us briefly recall the general features of the singular points denoting by λ_1 and λ_2 the two solutions of Eq. (16) [21]. If λ_1 and λ_2 are both real and negative, then all trajectories approach the origin as $t \to +\infty$ and the point (x_0, y_0) is a stable node. Conversely, if λ_1 and λ_2 are real and positive, then all trajectories move away from (x_0, y_0) as $t \to \infty$ and the point is unstable node. Also, if λ_1 and λ_2 are real but λ_1 is positive and λ_2 is negative, then the point is a saddle point; the trajectories approach the origin in the direction of the eigenvector associated to the eigenvalue λ_2 and moves away in the direction of the eigenvector associated to the eigenvalue λ_1. When the roots are pure imaginary then the vector $\mathbf{X}$ represents closed orbit and the singular point is a center. If λ_1 and λ_2 are complex with nonzero real part, then the singular point is a spiral point. When $\mathrm{Re}\,\lambda_{1,2} < 0$ then $X \to 0$ as $t \to +\infty$ and the singular point is a stable spiral point, conversely, when $\mathrm{Re}\,\lambda_{1,2} > 0$ the singular point is an unstable spiral point.

Let us check if these statements are fulfilled in our system (14). First, we can find that relations (15) vanish in the intersection point of the two curves

$$(1-\sigma)(2\sigma - 1 - 4\Delta\alpha(2-\alpha)) = (2\sigma - 1)(4\Delta\alpha - 1 + \sigma),$$
$$(1-\sigma)(2\sigma - 1 - 4\Delta\alpha(4-5\alpha)) = \left(4\Delta\alpha^2 + 1 - \sigma\right)(2\sigma - 1 - 8\Delta(1-2\alpha)).$$

Second, by using the common analysis on the system (14) that is a smooth map: $f_i : \Re \times \Re \to \Re$. We can found the following set of fixed points:

1. $x = 0,\ y = 0,$
2. $x = 0,\ y = \pm\sqrt{-a_1/a_3}$ if $-a_1/a_3 > 0,$
3. $x = \mp\sqrt{-b_1/b_3},\ y = 0$ if $-b_1/b_3 > 0,$
4. x and y values that satisfy simultaneously the following two system of equations. e.g. the intersections of the curves $a_1 + a_2x^2 + a_3y^2 = 0,\ b_1 + b_2y^2 + b_3x^2 = 0.$

The analysis was done by combining the analytic methods and computer simulations using the INTEGRA software developed in the faculty of sciences of the UNAM.

3.1. Linearization near the origin (0,0)

The eigenvalues of the linear system associated to the origin $(0,0)$ are $\pm\sqrt{a_1b_1}$. These values are real if $a_1b_1 \geqslant 0$, and pure imaginaries otherwise. In the first case the origin is a saddle point. Then, we have

$$a_1b_1 = -4\left(\sigma^2 - (4\Delta + 2)\sigma + 1\right).$$

Now if $a_1b_1 \geqslant 0$ then $\sigma^2 - (4\Delta + 2)\sigma + 1 \leqslant 0$. For $a_1b_1 = 0$ one has $\sigma = 2\left(\Delta \pm \sqrt{\Delta^2 + \Delta}\right) + 1$. The negative value of the square root we discard since $\sigma \leqslant 1$. Only the positive sign of the square root is possible. Let us observe the graphics of the function $f(\sigma) = \sigma^2 - (4\Delta + 2)\sigma + 1$. For the segment $2\left(\Delta - \sqrt{\Delta^2 + \Delta}\right) + 1 < \sigma < 1$, the theorem of Hartmann assures that the equilibrium is reached in the origin and this point is a saddle fixed point. Then the local behavior of the linearized system is similar to the original nonlinear dynamical system because we have nonpure imaginary eigenvalues. In the other case $a_1b_1 \leqslant 0$, $0 < \sigma < 2\left(\Delta - \sqrt{\Delta^2 + \Delta}\right) + 1$, the computer simulations show us that the origin is a center. We have here the typical linear oscillations. Then, in general case near the singular point $(0,0)$, for small ξ and η the linear oscillations have the dispersion relation

$$\frac{w^2}{\kappa^2} = 4\left(4\Delta\sigma - (1-\sigma)^2\right) \tag{17}$$

This expression is used in the next section to obtain the curve $\Delta = \left(1-\sigma^2\right)^2/4\sigma$ that reminds us of (qualitatively) the singular points of first- and second-order phase transition in the parametric space (σ, Δ).

3.2. Equilibriums along the axis $y : x = 0,\ y = \pm\sqrt{-a_1/a_3}$

These equilibriums exist if and only if

$$\frac{-a_1}{a_3} = \frac{\sigma - 1}{2\sigma - 1} > 0. \tag{18}$$

As $\sigma - 1 < 0$ then relation (18) holds if $\sigma < 1/2$. It is easy to note that if $\sigma \to 1/2$, then $-a_1/a_3 \to \infty$ and the equilibrium exists whether $0 < \sigma < 1/2$. The linearization around these points has the characteristic matrix

$$\begin{pmatrix} 0 & -2a_1 \\ \frac{b_1 a_3 - a_1 b_2}{a_3} & 0 \end{pmatrix}$$

with the expression for the eigenvalues $\lambda^2 + (2a_1/a_3)(b_1 a_3 - a_1 b_2) = 0$. It can be shown that $(b_1 a_3 - a_1 b_2) = 3\alpha\sigma - 2(1 - \sigma)$. In this case the roots are real or pure imaginaries. The real eigenvalues satisfy $\sigma > 1/(1.5\alpha + 1)$. Since the eigenvalues $\lambda_1 > 0$ and $\lambda_2 < 0$ we have the typical saddle point. As is known the saddle point is an unstable singular point for any direction of time. For the other case $\sigma < 1/(1.5\alpha + 1)$ the singular points are only centers. The equation $\sigma = 1/(1.5\alpha + 1)$ represents in the parametric space a curve of bifurcation from saddle singular points to centers. We can observe that this drastic change suggests a transformation of the dynamical property of the system from a nonequilibrium state to stable or quasi-stable states and vice versa. The computer simulations also indicate us that these singular points are centers and saddles.

3.3. Equilibriums along the axis $x : y = 0,\ x = \pm\sqrt{-b_1/b_3}$

Here the necessary condition for the existence of singular points is $-b_1/b_3 > 0$, being $-b_1 = 2(\sigma - 1 + 4\alpha^2\Delta)$ and $b_3 = -2(2\sigma - 1 - 4\Delta\alpha(4 - 5\alpha))$. To this end, both these expressions should have the same sign.

As we can see the curve $b_1(\sigma, \Delta) = 0$ or

$$\Delta = \frac{(1 - \sigma)^2}{4\sigma}$$

is exactly the same curve of bifurcations between saddles and centers and was reported above in Section 3.1. The equation $b_3 = 0$ gets the curve

$$\Delta = \frac{(2\sigma - 1)(1 - \sigma)}{16\sqrt{\sigma - \sigma^2} - 20\sigma}$$

that has a vertical asymptote when $\sigma = 3/8$. It is easy to check also that the eigenvalues for this case should support two types of singular points: saddles and centers.

The other possibility when the singular points satisfy the relations $x^2 \neq 0$ and $y^2 \neq 0$ does not give new information. For this specific case the fixed points in the plane (x, y) do not have pure real values. Let us check this statement. First of all, relation (15) transforms to

$$\begin{aligned} x_0 &= \pm\frac{1}{2}\sqrt{\frac{(\alpha\sigma + 2\sigma - 2)}{2\alpha\Delta(1 - 2\alpha)(\alpha - 2)}} = \pm\frac{1}{2}\sqrt{\frac{f}{g}}, \\ y_0 &= \pm\frac{1}{2}\sqrt{\frac{(-\alpha\sigma - 2\sigma + 2 + 8\alpha^2\Delta(1 - 2\alpha))}{2\alpha\Delta(1 - 2\alpha)(\alpha - 2)}} = \pm\frac{1}{2}\sqrt{\frac{h}{g}}. \end{aligned} \tag{19}$$

Second, suppose the denominator of each of the above relations (Eq. (19)) is positive. This implies that the numerator of x_0^2 and that of y_0^2 has to be positive too. A simple analysis shows that if $g \geqslant 0$ then $1/2 < \alpha < 2$ and for $f > 0$ one gets $\alpha > (\sqrt{2})^{1/3}$. Making the similar calculations in the second relation, one gets that if $h > 0$ then $\alpha < 1/2$. We have in these case no intersection between the regions of values for the parameter α in both numerators. This means that if we have real values (say) for $x_0^2 \neq 0$, the values of $y_0^2 < 0$ and vice versa.

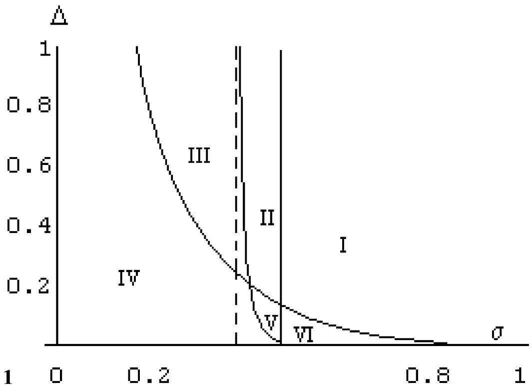
Consider now the case when the common denominator has negative sign. In this case the line of reasoning is similar and there were not encountered any intersections between the regions of validity of the parameter α in both numerators. This means that at the same time, simultaneously, we are unable to have positive values for the both point x_0^2, y_0^2, if one of them is fixed. Then, we disregard these possibilities. Concluding, the points of equilibriums are only those that we have reported above disregarding the values $x^2 \neq 0$ and $y^2 \neq 0$.

4. Bifurcations and phase transitions

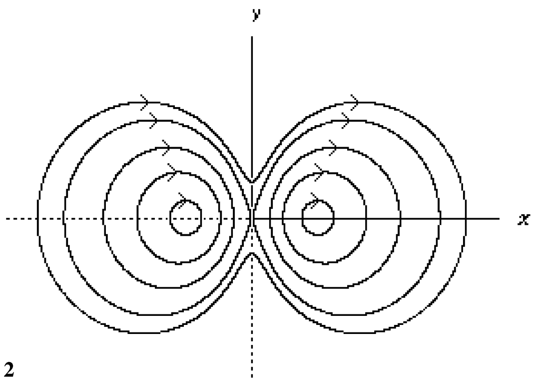
As is well known bifurcation is a change of topological structure of partitioning of the phase space of a dynamical systems into trajectories which is caused by small variation of system's parameters. We have the parametric space (σ, Δ) divided into six regions (see Fig. 1). The phase portrait obtained with the help of computer simulations shows us the following pictures. In regions I–III the origin is a saddle and in the regions IV–VI it is a center. In region I we have two centers in the axis x and one saddle at the origin (see Fig. 2). In region II there are two centers in the axis x, two in the axis y and one saddle at the origin (Fig. 3). In region III we have one saddle at the origin and two centers in the axis y (Fig. 4). In region IV we have centers in the axis y and at the origin too and there are saddles in the axis x. It is easy to note that saddles are connected by their separatrices (Fig. 5). In region V one has centers in the axis y and at the origin too (Fig. 6) and in region VI the only fixed point is the origin and it is a center (Fig. 7).

According to the computer simulations the bifurcations are as following. If one goes from region I to II, one finds two centers in the axis y that are coming from the infinity $\pm\infty$ and tends to the origin in accordance with the displacement of the point (σ, Δ) that turn on away from the line $\sigma = 1/2$. From region II to III the singular points along the axis x are departing away from each other tending to $\pm\infty$ and disappearing. From the region III to IV the origin changes its form from saddle to a center point. From region IV to V the points of the saddles in the axis x are going off the origin. From region V to VI the centers of the axis y are departing away according to the situation that they are approaching those points in the parametric space at the line $\sigma = 1/2$ and finally they disappear. Some complex bifurcations appear when one goes from the region I and pass to the region VI and from region II to V. In these cases the origin that is a saddle point transforms to a center and the previous centers disappear.

The phase transitions in the parametric space (σ, Δ) of this quasispin model were analyzed in previous works (PW) [16,19]. It was found that mainly two types of transition could occur in this model: the first and the second order phase transitions. These changes of phases take place when varying the values of the



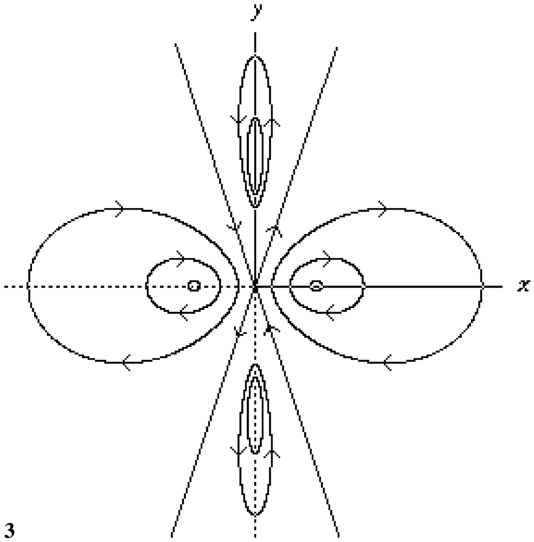
1



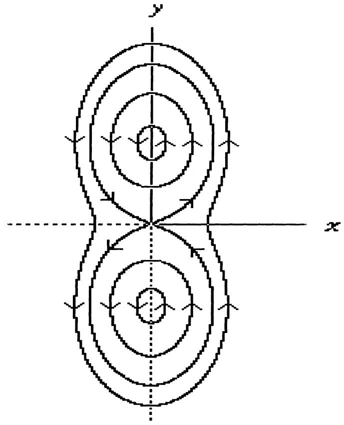
2

Fig. 1. The parametric phase space is divided into various regions depending on the behavior of its singular points of the nonlinear dynamical system (14). The straight line when $\sigma = 1/2$ is outside of our analysis since it represents an integrable system.

Fig. 2. The phase trajectories near the singular points in the region I. The phase trajectories depict saddles and are very similar to those where kinks appear in the nonrelativistic version of this model.



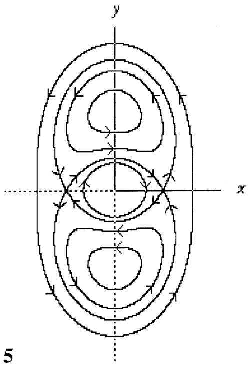
3



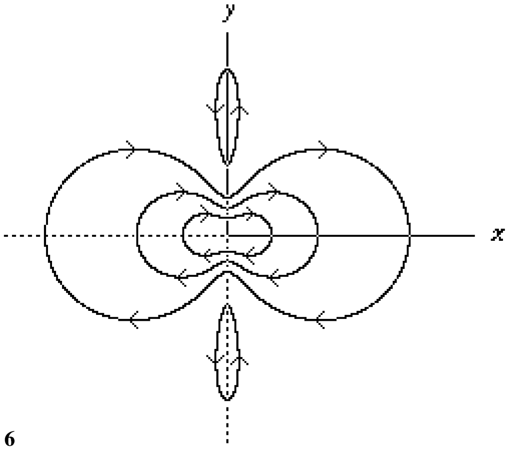
4

Fig. 3. In the region II one can observe that the singular points are centers and a saddle.

Fig. 4. The singular points in the region III are once again centers and one saddle near the $(0,0)$ point. The picture is the same as in the region I but rotated in an $\pi/2$ angle.



5



6

Fig. 5. The phase trajectories are more complicated in the region IV but they conserve some saddles and centers. In this region bubble solitons and plane waves could "live".

Fig. 6. The region V is the smallest of all regions we have here predominantly centers and one saddle.

relevant parameters that in some sense should be called "temperatures". The responsible for such kinds of transitions are the bubble and kink soliton solutions that "live" in the system. The kink soliton excitation in the ordered phase is responsible for the second-order phase transition to the disordered phase. The bubble solitons correspond to a localized spatial region of a vanishing field inside an ordered phase represented by the ground state. The condensation of bubbles generates the first-order phase transitions. On the other hand, as is known, the mere appearance of saddle singular points underlies the existence of kinks and chaos. It was demonstrated [14] that the bubble solitons could appear also from saddle singular points.

Let us examine to what extent the behavior near the singular points obtained here is relevant for studying the parametric phase transitions. By extrapolating and comparing our results with the corresponding ones of PW we can make several important conclusions. In fact, when translating from III region

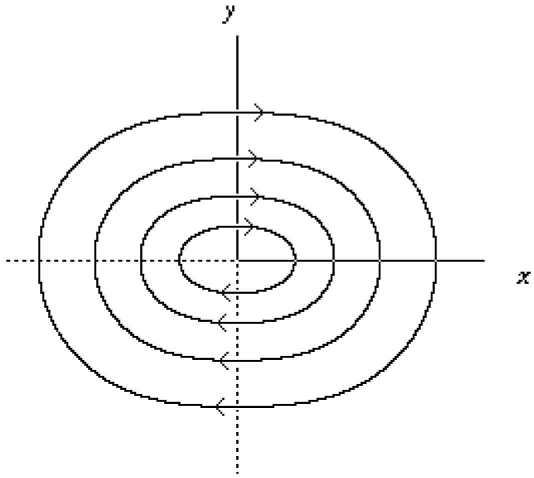


Fig. 7. The phase trajectories in the region VI describe periodic or quasi periodic oscillations. When passing from region I to VI one (or vice versa), the second-order phase transition is observed.

to the IV one (see Fig. 1) we can notice that a first-order phase transition could occur. This only because of the change in the number of saddles and centers. This means subsequently that we have two phases with different probabilities to exist. We could apply the similar reasoning to the passing between sectors IV and V. In the region IV bubbles could also appear because of the existence of saddles. In the region V there are predominantly centers and also there are saddles. This means that in both these sectors the possible oscillations are of quasiperiodic or periodic type in addition to the bubble solitons. When we check the transition from VI to the I region a crude restructuring of the phase space is observed. In region I we have similar phase trajectories that in some sense resemble the phase trajectories of kinks and in the region VI we have only a center. This last region could contain the wave planes. In this case of passing from sector I to sector VI we have second-order phase transition.

Resuming, we can say that by varying the parameters, principally the parameter Δ (fixing σ), one could find that saddles and centers could emerge and disappear when passing from one region to another. The regions which could contain bubbles and kinks in the present paper approximately converge to the regions of the parametric space of PW and in some sense they complement each other.

5. Conclusions

We have used here the results obtained in various works regarding the GCS and apply them to the quasispin model of the quantum Hamiltonian obtained by Masperi. As is well known the GCS are a good (sometimes appropriate) tool for averaging. We studied the simplest dynamical system treated by GCS method for the lattice model of the phi-six theory and now are able to catch essential features concerning bifurcations of the singular points that implies phase transitions in the parametric phase space of the model.

As can be inferred from the computer simulations and the analytic calculations realized on the lattice nonlinear equation, the reduced version of this discrete model belongs to the conservative or to the so called Hamiltonian system. This reduced version does not embrace, of course, all the rich dynamics that the system (10) should possess possibly in the second or more approximation. Instead, we have observed that by analyzing this relatively simple version, one can get a suitable picture of the parametric phase transition in which the solitonic bubbles and kinks are responsible. By studying the dynamics (specially its singular points) of this simple model one can predict important conclusions around the role of solitonic bubbles and kinks.

All the possibilities of singular points of this system are shown in Figs. 2–7. As is seen from the results the nondegenerate fixed points (that means no eigenvalue of the linearization is zero) are either center or saddle points. The right implication of the singular points in the analysis of phase transition can be resumed as following. The saddle points obtained here underlie the existence of both bubble and kink solitons and as it is obvious the centers are responsible for linear waves. The first-order phase transition occurs when the saddles and centers transform to another configuration with saddle and center singular points. The second-order phase transition appears when a center bifurcates to saddle points and to other centers or these points transform to only one center. Similarly, of course the account of bifurcations into the phase transition could be done in the language of Poincare indexes and topological equivalence of phase spaces. All these reasonings support the statement of the meaning of first and second-order phase transition: As usual, we can consider first-order phase transition only those of transitions in which "below " and "above" the critical points (in σ, Δ space) both phases exist simultaneously (though) with different probabilities. Second-order phase transitions are those transitions in which "below" and "above" the critical points only one phase "lives" in each sector.

Acknowledgements

Authors are indebted to Professor V.G. Makhankov for constant support discussions and also to Dr. H. Carrillo who provided us with the software INTEGRA developed in the Faculty of Science of the UNAM, Mexico. In addition we thank Osmara Vega for revising the manuscript. This work was support in part by the project 1336/98 UAEM.

References

[1] Klauder JR, Sudarshan ECG. Fundamentals of quantum optics. New York: Benjamin, 1968.
[2] Glauber RJ. Phys. Rev. 1963;130:2529.
[3] Glauber RJ. Phys. Rev. 1963;131:2766.
[4] Aragone C, Chalbaud E, Salamó S. J. Math. Phys. 1963;17:176.
[5] Lieb EH. Commun. Math. Phys. 1973;31:327.
[6] Kuratsuji H, Susuki T. J. Math. Phys. 1980;21:472–80.
[7] Zhang WM, Feng DH, Gilmore R. Rev. Mod. Phys. 1990;62:867.
[8] Perelomov A. Generalized coherent states and their applications. New York: Springer, 1986.
[9] Friedberg R, Lee TD, Sirlin A. Nucl. Phys. B 1976;115:32.
[10] Kartavenko VG. Sov. J. Nucl. Phys. 1984;40:240.
[11] Gagnon L, Winternitz P. Phys. Rev. A 1989;39:296.
[12] Pushkarov D, ICTP preprint LAMP, 1995:/95/5.
[13] Agüero M, Espinoza G. Phys. Lett. A 1993;182:294.
[14] Makhankov VG. Soliton phenomenology. Dordrecht: Kluwer Academic Publishers, 1990.
[15] Makhankov VG, Agüero MA, Makhankov AV. J. Phys. A. 1996;29:3005–19.
[16] Agüero M, Alvarado R, Frias G. Physica D 1998;123:448–59.
[17] Landau LD, Lifshitz EM. Statistical physics. Moscow: Goztekhizd, 1981 [Chapter XIV].
[18] Bruce AD, Cowley RA. Structural phase transitions. London: Taylor and Francis Ltd, 1981.
[19] Boyanovsky D, Masperi L. Phys. Rev. D 1980;21(n6):1550.
[20] Masperi L. Phys. Rev. D 1990;41:3263.
[21] Sagdeev RZ, Usikov DA, Zaslavsky GM. Nonlinear physics. From the pendulum to turbulence and chaos. New York: Harwood Academic Publishers, 1988.