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On the firing maps of a general class of forced integrate and fire neurons ☆

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Abstract

Integrate and fire processes are fundamental mechanisms causing excitable and oscillatory behavior. Van der Pol [Philos. Mag. (7) 2 (11) (1926) 978] studied oscillations caused by these processes, which he called ‘relaxation oscillations’ and pointed out their relevance, not only to engineering, but also to the understanding of biological phenomena [Acta Med. Scand. Suppl. CVIII (108) (1940) 76], like cardiac rhythms and arrhythmias. The complex behavior of externally stimulated integrate and fire oscillators has motivated the study of simplified models whose dynamics are determined by iterations of ‘firing circle maps’ that can be studied in terms of Poincaré’s rotation theory [Chaos 1 (1991) 20; Chaos 1 (1991) 13; SIAM J. Appl. Math. 41 (3) (1981) 503]. In order to apply this theory to understand the responses and bifurcation patterns of forced systems, it is fundamental to determine the regions in parameter space where the different regularity properties (e.g., continuity and injectivity) of the firing maps are satisfied. Methods for carrying out this regularity analysis for linear systems, have been devised and the response of integrate and fire neurons (with linear accumulation) to a cyclic input has been analyzed [SIAM J. Appl. Math. 41 (3) (1981) 503]. In this paper we are concerned with the most general class of forced integrate and fire systems, modelled by one first-order differential equation. Using qualitative analysis we prove theorems on which we base a new method of regularity analysis of the firing map, that, contrasting with methods previously reported in the literature, does not requires analytic knowledge of the solutions of the differential equation and therefore it is also applicable to non-linear integrate and fire systems. To illustrate this new methodology, we apply it to determine the regularity regions of a non-linear example whose firing maps undergo bifurcations that were unknown for the previously studied linear systems. © 2001 Published by Elsevier Science Inc.

Keywords: Forced oscillators; Integrate and fire neurons; Phase locking; Synchronization; Cardiac rhythms; Circle maps

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1. Integrate and fire systems models

Integrate and fire processes are studied in a wide variety of scientific domains [2–5,7,8]. Their universal importance comes from the fact that they are fundamental causes of excitable and non-linear oscillatory behavior. These processes are characterized by the presence of at least one magnitude (state variable), $x(t)$, that tends to accumulate (*integration process*), as well as some kind of mechanism that triggers a sudden discharge of the system (*firing process*), when the state variable $x(t)$ reaches a (normalized) threshold value $x = 1$. This discharge involves the relaxation of the state variable $x(t)$ to a (normalized) rest value $x = 0$. The time series of the discharges are called *fire sequences* and typically obey a periodic pattern. van der Pol emphasized the importance of these oscillations which he called *relaxation oscillations* [1]. They constitute an important prototype of autonomous (self sustained) oscillations. A typical example of a relaxation oscillator is an RC electrical circuit with a Neon bulb whose electrical discharges are depicted in Fig. 1. Since van der Pol, the study of the response types of these oscillators to a periodic stimulation has been a matter of wide interest among scientist and engineers and has led to sophisticated studies of synchronization (phase locking). Today the modelling of forced systems constitutes a main issue in the theory of non-linear oscillations.

In terms of differential equations, a relaxation oscillator is usually modelled by a system of (at least) two first-order differential equations that describe these processes, relaxation and accumulation, that occur in different scales of time. However, a strong simplification of the model is accomplished under the consideration that the fast relaxation process, happens instantaneously. Under this assumption the forced oscillator can be modelled with just one differential equation using an ingenious artifice [5]: the state variable $x(t)$ of the system is assumed to follow the first-order differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(t,x) \tag{1}$$

until $x(t)$ reaches the *threshold value* $(x = 1)$ at a time ξ, called the *firing time*, then the system undergoes a jump (discontinuous change) that resets the magnitude x to its *rest value* $(x = 0)$ and from there on, the system follows another solution of the differential equation, satisfying $x(0) = \xi$, until the threshold is met again and the process repeats itself. Thus, in this model, the evolution of the state variable x is considered to be described by a piecewise differentiable function $x(t)$, with jump discontinuities at the firing times, where it satisfies the *firing condition*

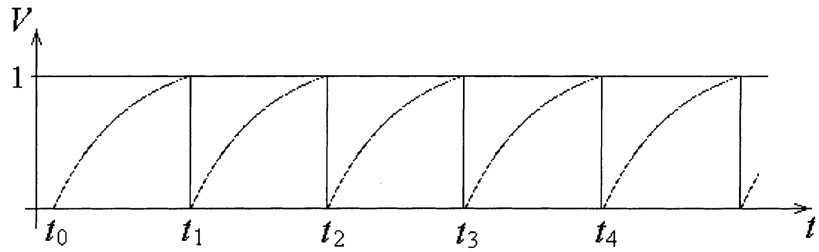

Fig. 1. Integrate and fire oscillations of the Neon bulb electrical circuit.

$$\lim_{t\to s^+} x(s) = 0 \quad \text{if } x(s) = 1. \tag{2}$$

Naturally, models for which the differential equation (1) is autonomous, correspond to free (unforced) systems that oscillates in a periodic way and nonautonomous differential equation models like (1) correspond to externally stimulated (forced) systems. Periodically forced systems are modelled with periodic right hand sides of this equation.

2. Firing maps

This paper is focused on integrate and fire systems models that are given by the differential equation (1) and the firing condition (2). The dynamics of these systems are conveniently described by a *firing map* $a : \mathbb{R} \to \mathbb{R}$. This is the map that for any given time t (for which the system fires), gives us the next firing time $a(t)$.

For convenience of the mathematical analysis, it will be assumed that the function $F : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is analytic and Ω is a region that contains the strip

$$\{ -\infty < t < +\infty,\ 0 \leqslant x \leqslant 1\}.$$

Following the usual notation, for any given (τ, η) in Ω, we will use $x(t;\tau,\eta)$ to denote the solution of Eq. (1) which satisfies the initial condition $x(\tau) = \eta$. We will say that the system *fires from the initial condition* $\tau \in \mathbb{R}$ if there is $t' > \tau$ such that $x(t';\tau,0) = 1$. In this case, the equation $x(t;\tau,0) = 1$, must have a minimal solution and the *firing map, a*, is well defined by the rule

$$a(\tau) = \min\{t > \tau : x(t;\tau,0) = 1\}.$$

The natural domain of this map is the set (D_a) of all times, τ, from which the system fires:

$$D_a = \left\{\tau \in \mathbb{R} : \exists t' > \tau \text{ such that } x(t';\tau,0) = 1\right\}.$$

The time $a(\tau)$ is called the (next) *firing time* from τ. The systems *firing sequence* $\{t_n\}$, starting from the time τ, is given, recursively, by

$$t_0 = \tau,$$
$$t_n = \min\{t > t_{n-1} : x(t;t_{n-1},0) = 1\} \quad \forall n \geqslant 1.$$

or, which is the same:

$$t_0 = \tau,$$
$$t_{n+1} = a(t_n).$$

Example (*Free oscillations*). When the function $F(t,x)$ is independent of t and $D_a \neq \emptyset$ (which implies that $F(x) \neq 0$ for $x \in [0,1]$), then (1) with the firing condition (2) models an autonomous integrate and fire oscillator for which $a(\tau) = \tau + T$, with T given by

$$T = \int_0^1 \frac{\mathrm{d}x}{F(x)}$$

and therefore the firing sequences are given by

$$t_{n+1} = \tau + nT.$$

When the system fires from all initial time τ ($D_a = \mathbb{R}$), we have that all the possible firing sequences $\{t_n\}$ are well defined for all $n \in \mathbb{N}$ and therefore these sequences are the orbits in the real line of the discrete semi-dynamical system in $\mathbb{R}$ generated by the iterations of the firing map a. Furthermore, when the firing map is injective the sequences are also reversible and will actually be the orbits of a dynamical system in the line. In the case when D_a is a proper subset of $\mathbb{R}$ we still have a semi-dynamical system (or a dynamical system if a is injective) restricted to D_a, if all firing times are in D_a, that is to say if $a(D_a) \subset D_a$. It has been found in [9] that even for linear systems like the KHR model described below, there are some parameter values for which the image of D_a is not necessarily contained in D_a and hence for some initial conditions the system produce only finite sequences of discharges before the firing process ends.

3. Synchronization

Owing to its scientific relevance, the periodically forced case deserves special attention. In such a model the function $F(t, x)$ is periodic in t ($F(t + T, x) = F(t, x)$) and, measuring the time in units of this forcing period we can normalize the period to the value $T = 1$. One of the main issues to investigate, in relation with a periodically forced non-linear oscillator, is whether the systems response would be also periodic and, if that is so, whether it would be synchronized with respect to the forcing or not. In this context, synchronized responses do not imply that the period T_{R} of the system response is equal to the period T_{F} of the external forcing, but more generally means that they are rationally related: $T_{\mathrm{R}} = p/q\, T_{\mathrm{F}}$. This phenomenum is also known as *phase or mode locking*.

To study the synchronicity properties of integrate and fire models, it is convenient to relate the firing times to the sequence of cyclic phases of the forcing process. Therefore, instead of dealing exclusively with the firing sequences $\{t_n\}$, it is convenient to consider the *firing phase sequences* $\{s_n\}$:

$$s_n = t_n \bmod 1 \quad \forall n \geqslant 0.$$

These phase sequences tell us in which phase, relatively to the periodic forcing cycle, the system is firing. In [5] Keener, Hoppensteadt and Rinzel (KHR) pointed out the remarkable fact that these phase sequences are determined by the iterations of a circle map: since for any solution $x(t)$ of Eq. (1), the function $x(t + 1)$ is also a solution, it follows that the firing map satisfies the circle mapping property

$$a(t + 1) = a(t) + 1.$$

Therefore, (when $D_a = \mathbb{R}$) the firing map a is the lift of a degree one circle map, called the *firing phase map*, or just the *phase map,* of the system. The synchronization properties of the forced oscillator are encoded in the dynamics of this circle map: e.g., the system fires q times during p cycles of the forcing ($q : p$ *synchronization*) if and only if the firing phase map has a periodic attractor of *period* q and *wrapping* number p, i.e., there is a time t such that $a^q(t) = t + p$.

Commonly in applications the forced oscillator involves parameters and therefore Eq. (1) takes the form

$$\frac{dx}{dt} = F(t, x, \lambda), \tag{3}$$

where $\lambda = (\lambda_1, \ldots, \lambda_k)$. The regions in parameter space where the firing phase map has a periodic attractor, are called *synchronization zones* or *Arnold tongues*. As the system crosses these tongues the firing map a_λ undergoes bifurcations and the synchronization properties of the system change. The theory of circle maps is a valuable tool to investigate these bifurcations and therefore the structure of the synchronization zones in parameter space (bifurcation diagram).

4. Regularity regions

The regularity properties (continuity and injectivity) of the model (3) firing map, $a_\lambda(t)$, give us information about the synchronization zones patterns in parameter space. According to the *regularity type* of the map $a_\lambda(t)$, there would be four *regularity regions* of interest, in the parameter space:

(I) region where $a_\lambda(t)$ is a homeomorphism;
(II) region where $a_\lambda(t)$ is continuous;
(III) region where $a_\lambda(t)$ is injective;
(IV) region where $a_\lambda(t)$ is neither injective nor continuous.

In the region I, the limit

$$\rho(a_\lambda) = \lim_{n\to\infty} \frac{a_\lambda^n(t)}{n} \mod 1, \tag{4}$$

known as the Poincaré's *rotation number* of the firing map, exists, and is independent of t. The rotation number $\rho(a_\lambda)$ is a useful tool to guarantee the existence of periodic orbits of the firing map a_λ [10]: when $\rho(a_\lambda)$ is rational (p/q), the set of periodic orbits of the map is not empty and all orbits have period q and wrapping number p. Thus, in this regularity region, the system cannot have more than one type of synchronization (i.e., the synchronization zones cannot intersect each other in region I). On the other hand [11], if the rotation number of the phase map is not a rational number, then all its orbits are aperiodic and dense in the whole circle (or in a Cantor subset of it if the differentiability class of the phase map is not large enough) and there is no synchronization, but quasiperiodic behavior.

Outside of the regularity region I the rotation number is not defined, but a generalized rotation theory can be used: if the map is continuous but not monotone, we cannot ensure the existence of the limit (4) that defines the rotation number and when it exists it could depend on the value of the point, t, in which it is calculated; however, it is still possible to associate to each continuous circle map a *rotation interval* [12,13]. Various rotation interval properties are useful for synchronization studies; e.g., for each rational number p/q in the rotation interval, the firing phase map has at least one periodic orbit of period q [14]. Hence, periodic attractors (of different periods) might coexist (multistability phenomenon), meaning that the synchronization zones could intersect where the firing map is continuous but not injective (region II/region III) [15,16]. Another extension of the rotation theory has been provided by Keener [17]. He proved that a limit analogous to (4) is well defined for monotone maps with jump discontinuities and that, depending on whether ρ is rational or not, the map has periodic attractors or dense orbits in a Cantor set. It is an interesting fact that the firing maps a_λ, that arise from integrate and fire systems, could result discontinuous but, according to Propositions 8.11 and 8.12 proved below, their discontinuities must be of the jump type. Therefore, in the region III, the synchronization zones cannot intersect.

Geometrical models. This is a widely studied class of integrate and fire models for which the sawtooth curves are not solutions of a differential equation but just line segments obtained following a purely geometric procedure [3,4]. The phase dynamics of these periodically stimulated geometrical models are also determined by a circle map and some of these models have the advantage to be analytically tractable, so it is easy to calculate the boundaries of its regularity regions. In this class of geometrical models, the classical biparametric family of circle maps

$$a(t) = t + \alpha + \beta \sin(2\pi t) \bmod(1)$$

appears as a remarkable model of cardiac dynamics as well as neuron responses to cyclic stimulation [9,16].

Contrasting with the linear accumulation models (LAMs) of integrate and fire systems that we discuss in the following section, this classical model features the following property: the regularity region III is contained in region I (Fig. 2(b)). This is due to the obvious fact that, for all values of the parameters α and β, the elements of this family are continuous maps, and that these functions become non-injective when $\beta > \pi/2$. This paradigmatic example of circle maps has been widely studied by Arnold [18], and Herman [19] among other mathematicians interested in abstract and applied studies of ring dynamics. The integrate and fire differential models enrich the insight to the behavior of forced non-linear oscillators and provide new examples of multiparametric families of circle maps, which are not analytically expressible in terms of elementary functions and exhibit new dynamics. For example, region II is contained in region I (Fig. 2(c)) for the KHR model that we discuss in the following section, while the non-linear model that we analyze in Section 7 gives rise to a more general structure of regularity regions, like the one depicted in Fig. 2(a).

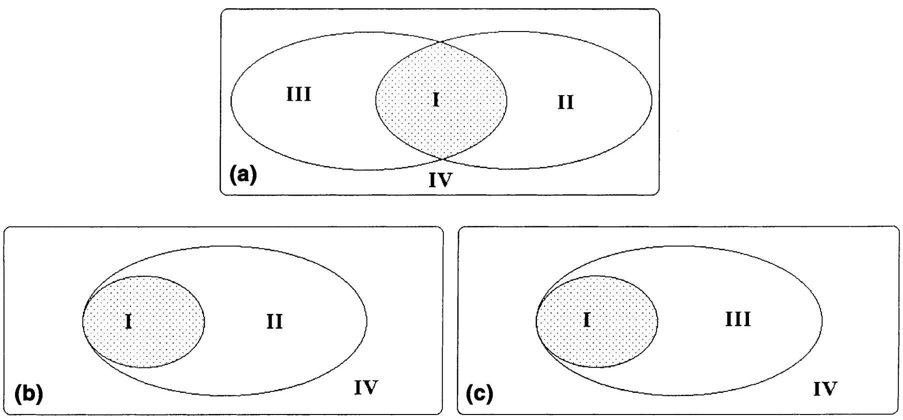

Fig. 2. Regularity regions in parameter space.

5. The linear accumulation model (LAM)

A general forced integrate and fire system with a linear rate of accumulation can be modeled with the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sigma x + g(t, \lambda), \tag{5}$$

where σ and $\lambda = (\lambda_1, \ldots, \lambda_k)$ represent the system's parameters. In spite of the system's linearity, the regularity properties of the firing maps cannot be directly verified. This is due to the fact that the firing maps of these integrate and fire systems are very elusive: even for harmonically forced LAMs (like the KHR model discussed below), for which the general solution of the differential equation can be obtained in terms of elementary functions, the corresponding firing maps are not analytically accessible.

The KHR model. Linear integrate and fire models have been investigated by several authors. In [5,7,20] a linear accumulation system with sinusoidal forcing ($\lambda = (S, H)$ and $g(t, \lambda) = S + H\sin(2\pi t)$) has been analyzed using a combination of analytic and numerical methods. Although it is not possible to obtain an analytic expression for the firing map, the general solution of the equation can be used to determine that the firing map a is implicitly determined by the equation

$$K(a(t)) = K(t) + \frac{\sigma}{S}\mathrm{e}^{\sigma t}, \tag{6}$$

where

$$K(t) = \mathrm{e}^{\sigma t}\left[1 - \frac{\sigma}{S} + \frac{H\sigma}{S\sqrt{\sigma^2 + 4\pi^2}}[\sigma\cos(2\pi t) + 2\pi\sin(2\pi t)]\right].$$

By means of a careful study of this equation, in [5], KHR determine the system's regularity regions in parameter space and combine this knowledge with circle maps rotation theory to reveal the synchronization properties of the system. Other authors [20,21] have applied analytical methods based on fire time ansatz, to find mode-locked solutions and carried out numerical simulations, in conjuction with Liapunov exponents calculations, to determine the dominant forms of synchronization. The KHR approach can be generalized [6] to study a general LAM of the form (5) with the aid of the following implicit equation for the firing map $a(t)$:

$$\int_t^{a(t)} \mathrm{e}^{\sigma s}(g(s, \lambda) - \sigma)\,\mathrm{d}s = \mathrm{e}^{\sigma t}. \tag{7}$$

The analysis carried out in [6,20] reveals the general validity of an interesting result about the structure of the linear accumulation systems regularity regions that was already observed for the KHR model: for any periodic forcing $g(t, \lambda)$, the systems region II must be contained in the region I. This accounts to say that linear accumulation systems cannot have continuous and non-injective firing maps. In this respect these forced oscillators are different from the geometrical sawtooth oscillators that give rise to the continuous and noninjective Arnold's firing maps, for which we have in contrast, that region III is contained in region I and therefore Poincare's rotation number theory, plus the Newhouse–Palis–Takens theory of rotation intervals applies to carry out the synchronization analysis. However, for the linear accumulation differential models, Keener's generalization of Poincare's rotation number for discontinuous (but injective) maps will also be required to analyze the system behavior in the (non-empty) region III.

Once that it has been observed that II = I is valid for all LAMs, the question arises of whether this is also true for the more general class of integrate and fire oscillators modelled by Eq. (1) with the jump condition (2). The scope of the linear methods is not broad enough to answer this type of questions since for non-linear systems an equation like (7) is not available.

The theory that we will discuss below uses qualitative methods of analysis to determine the boundaries, and hence the structure of the regularity regions partition for systems that do not have analytically accessible firing maps and which are not even analytically solvable. In Section 7 we use this theory to prove that there are non-linear models for which region III is not contained in region I, but have a richer structure that is qualitatively similar to the one shown in Fig. 2(a).

6. Regularity theorems

In this section we present the main theorems of this paper. They relate to the regularity properties of the firing maps (continuity and injectivity) generated by integrate and fire oscillators of the general class determined by (1) and jump condition (2). These theorems owe their usefulness to the fact that they characterize the properties of the system's firing map in terms of a priori conditions; that is to say, in terms of properties that can be verified directly from the differential equation (1) of the model, without requiring further knowledge of its solutions.

The theorems of this section apply when the interior of the domain of the firing map a is not empty.

Injectivity theorem. *The firing map a is injective in* $\text{int}(D_a)$ *if and only if* $F(t,0) \geqslant 0$ *for all* $t \in \text{int}(D_a)$.

Continuity theorem. *The firing map a is continuous in* $\text{int}(D_a)$ *if and only if* $F(t,1) \geqslant 0$ *for all t in an neighborhood of* $a(\tau), \tau$ *being any point in* $\text{int}(D_a)$.

These theorems are not restricted to equations that model periodically forced oscillators but, due to its scientific importance, we will make some remarks about the application of these theorems to determine the regularity regions of periodic systems dependent on parameters. When $F(t,x,\lambda)$ is periodic, the inequalities

$$F(t,0,\lambda) \geqslant 0,$$
$$F(t,1,\lambda) \geqslant 0$$

are satisfied for all $t \in R$, if and only if

$$I(\lambda) \equiv \text{Min}_t F(t,0,\lambda) \geqslant 0,$$
$$C(\lambda) \equiv \text{Min}_t F(t,1,\lambda) \geqslant 0$$

and therefore the sets in parameter space

$$I(\lambda) = 0,$$
$$C(\lambda) = 0$$

are the boundaries of the injectivity and continuity regions, respectively. These sets are (generally) codimension one manifolds in parameter space.

7. A non-linear model example

Before presenting the proof of the regularity theorems, in this section we illustrate their application to analyze the structure of the regularity regions in parameter space of the system (1) with the non-linear function

$$F(t, x, \lambda_1, \lambda_2) = (2 + \lambda_1 \sin(2\pi t) - x)(x + 1 + \lambda_2 \sin(2\pi t)).$$

We have that $F(t, 0, \lambda_1, \lambda_2) \geqslant 0$ for all t, if and only if

$$|\lambda_1| \leqslant 2 \quad \text{and} \quad |\lambda_2| \leqslant 1,$$

and that $F(t, 1, \lambda_1, \lambda_2) \geqslant 0$ for all t, if and only if

$$|\lambda_1| \leqslant 1 \quad \text{and} \quad |\lambda_2| \leqslant 2.$$

Therefore, the continuity and the injectivity regions are bounded by the regions depicted in Fig. 3, and the intersection of them determine the homeomorphism region.

The structure of the regularity regions of this system is rich enough to allow bifurcations that were impossible for linear accumulation systems and in fact produce a wider spectrum of bifurcations (transitions, from one regularity region to another) for the firing map a_λ.

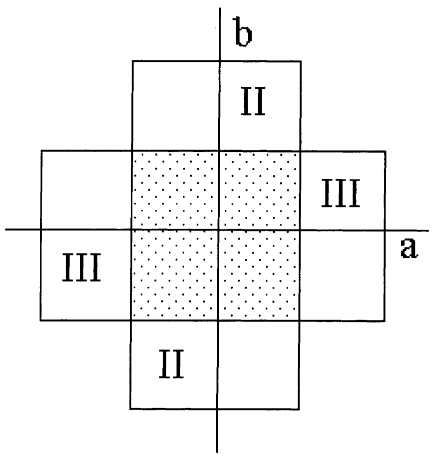

Fig. 3. Non-linear model: $F(t, x, \lambda_1, \lambda_2) = (2 + \lambda_1 \sin(2\pi t) - x)(x + 1 + \lambda_2 \sin(2\pi t))$. For $\lambda_1 = 0$ and $|\lambda_2| \to \infty$, the firing map first loses injectivity and thereafter loses continuity. For $\lambda_2 = 0$ and $|\lambda_1| \to \infty$, the firing map first loses continuity and thereafter loses injectivity. Starting from (0,0) and increasing $\lambda_1 = \lambda_2$, the firing map loses simultaneously injectivity and continuity at $(1, 1)$.

8. Proof of the regularity theorems

The proofs of the continuity and the injectivity theorems that we present here are fundamentally based on the analyticity hypothesis of the differential equation. Naturally this observation poses the question of whether these theorems would still be valid under weaker assumptions.

We start establishing some lemmas and definitions.

Lemma 8.1. *Assume that f is analytic in an open interval I and let us suppose that it vanishes in a convergent sequence, whose limit is contained in I. Then* $f \equiv 0$ *in I.*

Proof. Without lost of generality, we assume that there is a sequence $\{\tau_n\}_{n=1}^{\infty} \subset I$ such that $\tau_n < \tau_{n+1}$ and $\tau_n \to \tau^*$ as $n \to \infty$. We will conclude that $f \equiv 0$ in I, proving that $f(\tau)$ and all its derivatives vanishes at $\tau = \tau^*$. For this we inductively prove that, for each $k = 0, 1, \ldots$, there is an increasing sequence $\{\tau_n^k\}_{n=1}^{\infty} \subset I$, which tends to $\tau^*$and satisfies that $f^{(k)}(\tau_n^k) = 0$. Then, by the continuity of f and all its derivatives, it will follow that $f^{(k)}(\tau^*) = 0$ for all k.

The hypothesis validates the case $k = 0$. If for some $k \geqslant 0$ the induction hypothesis is satisfied, the mean value theorem implies that, there is τ_n^{k+1}, with $\tau_n^k < \tau_n^{k+1} < \tau_{n+1}^k$, such that

$$f^{(k)}(\tau_{n+1}^k) - f^{(k)}(\tau_n^k) = f^{(k+1)}(\tau_n^{k+1})(\tau_{n+1}^k - \tau_n^k).$$

Then $f^{(k+1)}(\tau_n^{k+1}) = 0$ and obviously $\tau_n^{k+1} \to \tau^*$. □

We say that an analytic function f is *strictly monotonic* in τ, if $f'(\tau) \neq 0$.

Proposition 8.2. *Nonconstant analytic functions assume their values isolatedly. That is to say, if f is analytic in an neighborhood of* τ_0*, and* $f(\tau_0) = c$*, then,* $f(\tau) \neq c$ *in a punctured neighborhood of* τ_0*. Also, for r small enough, f approaches strictly monotonically the value c, from the right (for* $t \in (\tau_0 - r, \tau_0]$*) and from the left (for* $t \in [\tau_0, \tau_0 + r)$*).*

Proof. Lemma 8.1 implies that $f(\tau) - c$ cannot vanishes arbitrarily close to (but not in) τ_0, so it assume their values isolatedly. For the second part of the corollary, the function f' cannot vanish arbitrarily close (to the left) of τ_0, otherwise, for all $r > 0$ there is $\tau_r \in (\tau_0 - r, \tau_0)$ such that $f'(\tau_r) = 0$, and then, by Lemma 8.1, f' would vanish in an interval that contains τ_0 which implies that f is constant, contradicting our hypothesis. □

Corollary 8.3. *In an neighborhood of a point* τ_0*, where the function f is analytic, we have the following alternatives:*

(i) *f is constant, or*
(ii) *f is strictly increasing, or*
(iii) *f is strictly decreasing, or*
(iv) *f has a local minimum at* τ_0*, or*
(v) *f has a local maximum at* τ_0*.*

Recall that Ω is the domain of definition of the analytic function F, which is a region that contains the strip

$\{-\infty < t < +\infty,\ 0 \leqslant x \leqslant 1\}$.

Lemma 8.4. *If $(\tau_0, \eta) \in \Omega$ is such that $F(\tau_0, \eta) > 0 (< 0)$, then there is an $r_1 > 0$ such that $x(t; \tau_0, \eta) < \eta\ (> \eta)$ in the interval $[\tau_0 - r_1, \tau_0)$, and, similarly, there is an $r_2 > 0$ such that $x(t; \tau_0, \eta) > \eta (< \eta)$ in the interval $(\tau_0, \tau_0 + r_2)$.*

Proof. For the solution $x(t) \equiv x(t; \tau_0, \eta)$ we have that

$$\dot{x}(\tau_0) = \lim_{\tau \to \tau_0} \frac{x(\tau) - \eta}{\tau - \tau_0} = F(\tau_0, 0).$$

Therefore, if $\tau < \tau_0$, sufficiently close to τ_0 and $F(\tau_0, 0) > 0$, then $x(\tau) < \eta$. The other cases are quite similar. □

Definition 8.1. We say that the solution $x(t)$ of (1) crosses ascendantly the line $x = k$ at the point (t_0, k) if:

(i) $x(t_0) = k$,

(ii) $F(t_0, k) \geqslant 0$,

(iii) for some $r > 0$, $0 < |t - t_0| < r$ implies that $F(t, k) > 0$.

We also say that $x(t)$ *crosses ascendantly at the point* (t_0, k).

Lemma 8.5. *Let (τ_0, η) be an interior point of Ω. If the solution $x(t; \tau_0, \eta)$ is not constant, there is $r > 0$ such that $F(\cdot, \eta) \neq 0$ and $F(\cdot, \eta)$ does not change its sign in neither of the intervals $(\tau_0 - r, \tau_0)$ and $(\tau_0, \tau_0 + r)$. Furthermore, in virtue of the Corollary 8.3, the following alternatives cover all the possible cases:*

(i) *if the solution $x(t; \tau_0, \eta)$ has a minimum (for $t \in (\xi, \tau_0]$) in τ_0, there is $r > 0$ such that $F(\tau, \eta) < 0$ for all $\tau \in (\tau_0 - r, \tau_0)$;*

(ii) *if the solution $x(t; \tau_0, \eta)$ has a maximum (for $t \in (\xi, \tau_0]$) in τ_0, there is $r > 0$ such that $F(\tau, \eta) > 0$ for all $\tau \in (\tau_0 - r, \tau_0)$;*

(iii) *if the solution $x(t; \tau_0, \eta)$ has a minimum (for $t \in [\tau_0, \xi)$) in τ_0, there is $r > 0$ such that $F(\tau, \eta) > 0$ for all $\tau \in (\tau_0, \tau_0 + r)$;*

(iv) *if the solution $x(t; \tau_0, \eta)$ has a maximum (for $t \in [\tau_0, \xi)$) in τ_0, there is $r > 0$ such that $F(\tau, \eta) < 0$ for all $\tau \in (\tau_0, \tau_0 + r)$.*

Proof. If $F(\cdot, \eta) = 0$ in an interval I around τ_0, the function $\bar{x}(t) \equiv \eta$, on I, is a solution of $\dot{x} = F(t, x)$; by unicity of solutions, $\bar{x}(t) = x(t; \tau_0, \eta)$ for all t in I, contradicting the hypothesis. Then $F(\cdot, \eta)$ is nonconstant and, by the Proposition 8.2, is different to zero in a punctured neighborhood of τ_0. Hence, by continuity, there is r such that $F(\cdot, \eta)$ does not change its sign in the interval $(\tau_0 - r, \tau_0)$.

We only prove assertion (i), the others being similar. Consider the interval $(\tau_0 - r, \tau_0)$ where $F(\cdot, \eta)$ does not change its sign and, contradicting the theorem thesis, suppose that $F(\cdot, \eta) > 0$ on it. We call Δ, the region of the (t, x)-plane bounded by the lines $x = \eta$, $t = \tau_0 - r$ and the curve $y = x(t; \tau_0, \eta)$ (see Fig. 4).

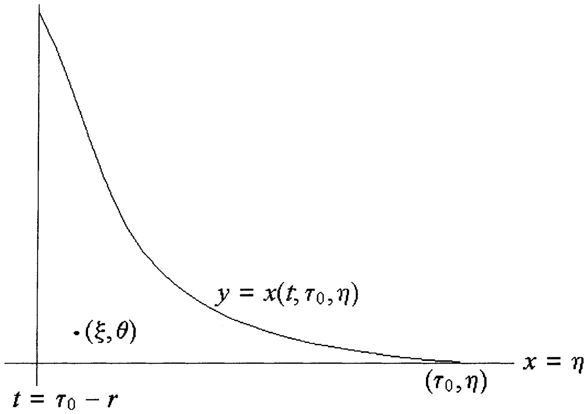

Fig. 4. Region Δ of the proof of Lemma 8.5.

We can chose r such that Δ is fully contained in Ω. If $(\xi, \theta) \in \text{int}(\Delta)$, for $t > \xi$ the solution $x(t; \xi, \theta)$ cannot meet the solution $x(t; \tau_0, \eta)$; neither it can reach the line $x = \eta$, because if it does so, we could use the Lemma 8.4 to prove that there would be a point where it has nonpositive slope. Since the graph of the solution $x(\cdot; \tau_0, \eta)$ has to be confined to the bounded region $\Delta \subset \Omega$, the interval $[\xi, \tau_0]$ has to be contained in the domain of this solution and $x(t; \xi, \theta) \to (\tau_0, \eta)$ as $t \to \tau_0$. Therefore, by uniqueness $x(\cdot; \xi, \theta) = x(\cdot; \tau_0, \eta)$ in $[\xi, \tau_0]$, contradicting that the point (ξ, θ) is an interior point of Δ. Then $F(\tau, \eta) < 0$ for all $\tau \in (\tau_0 - r, \tau_0)$. □

Corollary 8.6. *Let (τ, η) be an interior point of Ω the solution $x(\cdot; \tau, \eta)$ of Eq. (1) crosses ascendantly the line $x = \eta$ at (τ, η), if and only if there is $r > 0$ such that $x(t; \tau, \eta) < \eta$ for all $t \in (\tau - r, \tau]$ and $x(t; \tau, \eta) > \eta$ for all $t \in [\tau, \tau + r)$.*

Proof. Since $x(\cdot; \tau, \eta)$ crosses ascendantly at (τ, η), it is nonconstant. Assume $x(t; \tau, \eta) < \eta$ for all $t \in [\tau, \tau + r)$, for some $r > 0$. By Proposition 8.5 $F(t, \eta) < 0$ at the right of τ, contradicting that $x(\cdot; \tau, \eta)$ crosses ascendantly at (τ, η). For the converse, if the hypothesis are satisfied, then the Proposition 8.5 implies that $F(t, \eta) > 0$ in a punctured neighborhood of τ, so $x(\cdot; \tau, \eta)$ crosses ascendantly the line $x = \eta$ at (τ, η). □

Lemma 8.7. *Let (τ_0, η_0) a point in the domain, Ω, of F. If $x(t; \tau_0, \eta_0)$ crosses ascendantly the line $x = \eta_0$, at (τ_0, η_0), then for small enough ε, the map h*

$$h : (\tau_0 - \varepsilon, \tau_0 + \varepsilon) \to \mathbb{R}$$

defined by $h(\tau) = x(\tau_0; \tau, \eta_0)$ is a homeomorphism over its image (see Fig. 6) and $x(t; \tau, \eta_0) = x(t; \tau_0, h(\tau))$ for all $\tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)$.

Proof. By the continuity with respect initial conditions [22] there is $\varepsilon > 0$ such that $h(\tau)$ is well defined and continuous. Since there is an ascendant crossing, there is $\varepsilon > 0$ such that $F(\tau, \eta_0) > 0$ for $\tau \neq \tau_0$ and $\tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)$. If $h(\tau)$ were not injective, by uniqueness, there would be a solution that has several crosses (at least two) with the line $x = \eta_0$, in the ε-neighborhood τ_0. From the continuity of the solution, it follows that one of these crosses would occur with negative slope, which contradicts that $F(\tau, \eta_0) > 0$ for $\tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)$. Finally, since $x(\tau_0; \tau, \eta_0) = h(\tau) = x(\tau_0; \tau_0, h(\tau))$, by uniqueness, we have that $x(t; \tau, \eta_0) = x(t; \tau_0, h(\tau))$ for all $\tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)$. □

Proposition 8.8. *If the solution $x(t; \tau_0, 0)$ of Eq. (1) crosses ascendantly the line $x = 1$ at $(t, 1)$, for some $t \geqslant a(\tau_0)$ (but not necessarily at $(a(\tau_0), 1)$), then τ_0 is an interior point of D_a. The converse of this proposition is false, however, if τ_0 is an interior point of D_a, and, around the point $(\tau_0, 0)$ the solution $x(t; \tau_0, 0)$ match one of cases* (ii)–(iv) *of Corollary 8.3, then the solution $x(t; \tau_0, 0)$ crosses ascendantly the line $x = 1$ at $(t, 1)$, for some $t \geqslant a(\tau_0)$. If the solution $x(t; \tau_0, 0)$ reaches a maximum value at τ_0 (case* (v) *of Corollary 8.3), then τ_0 is always an interior point of D_a.*

Proof. Suppose the solution $x(t; \tau_0, 0)$ crosses ascendantly the line $x = 1$ at $(t_0, 1)$. Let $\varepsilon > 0$ such that $F(t, 1) > 0$ for $0 < |t - t_0| < \varepsilon$ and (using $(\tau_0, \eta_0) = (t_0, 1)$) consider the homeomorphism

$$h : (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathbb{R}$$

defined in Lemma 8.7. Since h^{-1} is continuous and $h(t_0) = 1$, there is an ε' such that the interval $(1 - \varepsilon', 1 + \varepsilon') \subset h(t_0 - \varepsilon, t_0 + \varepsilon)$. By continuity of solutions with respect to initial conditions, there is $\delta > 0$ such that if $|\tau - \tau_0| < \delta$, then $|x(t_0; \tau, 0) - x(t_0; \tau_0, 0)| = |x(t_0; \tau, 0) - 1| < \varepsilon'$. Therefore, $x(t_0; \tau, 0) \in h(t_0 - \varepsilon, t_0 + \varepsilon)$. Observe that $x(\cdot; \tau, 0) = x(\cdot; t_0, x(t_0; \tau, 0))$, by the other hand Lemma 8.7 implies that

$$x(\cdot; t_0, x(t_0; \tau, 0)) = x(\cdot; h^{-1}(x(t_0; \tau, 0)), 1),$$

therefore,

$$x(\cdot; \tau, 0) = x(\cdot; h^{-1}(x(t_0; \tau, 0)), 1),$$

hence

$$x(h^{-1}(x(t_0; \tau, 0)); \tau, 0) = 1$$

for all τ such that $|\tau - \tau_0| < \delta$. This proves that $\tau \in D_a$, thus τ_0 is an interior point of D_a.

Consider case (v) of Corollary 8.3, by uniqueness, $x(t; \tau, 0) \geqslant x(t; , 0)$ for τ in an neighborhood of τ_0, hence $\tau \in D_a$ and τ_0 is an interior point of D_a. Therefore, it is possible for a solution to have interior points τ_0 (of type (v)) for which the solution $x(t; \tau_0, 0)$ does not crosses ascendantly the line $x = 1$.

It remains to prove that for all interior points of types (ii)–(iv) of Corollary 8.3, the solution $x(t; \tau_0, 0)$ crosses ascendantly the line $x = 1$ at some $t \geqslant a(\tau_0)$.

We prove first that if $\tau_0 \in D_a$ and $x(t; \tau_0, 0) > 1$ for some $t > \tau_0$, then the solution $x(t; \tau_0, 0)$ crosses ascendantly the line $x = 1$ at some point $(\tau, 1)$ with $\tau > \tau_0$. From Proposition 8.2 it follows that, if there is $t > \tau_0$ such that $x(t; \tau_0, 0) > 1$, there are $t_1 < t_2$, with $\tau_0 < t_2$, such that

$$x(t_1; \tau_0, 0) < 1 < x(t_2; \tau_0, 0),$$

and the equation $x(t; \tau_0, 0) = 1$ has just one solution t^* in the interval $[t_1, t_2]$. Then, in the interval $[t_1, t_*]$ the solution must have a maximum at t^*. Similarly, in the interval $[t_*, t_2]$, a minimum must be attained at t^*. From Lemma 8.5, $F(t,1) > 0$ in a punctured neighborhood of t^* and hence $x(t;\tau_0,0)$ crosses ascendantly the line $x = 1$ at $(t^*,1)$.

Assume that τ_0 is an interior point of D_a, and, around the point $(\tau_0,0)$, the graph of $x(t;\tau_0,0)$ match case (ii) of Corollary 8.3 (cases (iii) and (iv) can be proved analogously). If the solution $x(t;\tau_0,0)$ does not cross ascendantly the line $x = 1$, at none $(t,1)$, with $t > a(\tau_0)$, then

$$\sup_{t \geqslant \tau_0} x(t;\tau_0,0) \leqslant 1.$$

This is due to the fact that if $\tau_0 \in D_a$ and $x(t;\tau_0,0) > 1$ for some $t > \tau_0$, then the solution $x(t;\tau_0,0)$ must cross ascendantly the line $x = 1$ at some point $(\tau,1)$ with $\tau > \tau_0$: from Proposition 8.2 it follows that, if there is $t > \tau_0$ such that $x(t;\tau_0,0) > 1$, there are $t_1 < t_2$, with $\tau_0 < t_2$, such that

$$x(t_1;\tau_0,0) < 1 < x(t_2;\tau_0,0),$$

and the equation $x(t;\tau_0,0) = 1$ has just one solution t^* in the interval $[t_1,t_2]$. Then, in the interval $[t_1,t_*]$ the solution must have a maximum at t^*. Similarly, in the interval $[t_*,t_2]$, a minimum must be attained at t^*. From Lemma 8.5, $F(t,1) > 0$ in a punctured neighborhood of t^* and hence $x(t;\tau_0,0)$ crosses ascendantly the line $x = 1$ at $(t^*,1)$.

Since we are assuming case (ii) of Corollary 8.3, around $(\tau_0,0)$, the solution $x(t;\tau_0,0)$ is strictly increasing. Then, in an interval of the form $[\tau_0,\tau_0+r]$, $x(t;\tau_0,0)$ attains a minimum at τ_0. By uniqueness, for $\tau \in (\tau_0,\tau_0+r]$, we have that $x(t;\tau,0) < x(t;\tau_0,0) \leqslant 1$ for all $t \geqslant \tau_0$. Hence $\tau \notin D_a$ and therefore τ_0 is a boundary point of D_a. □

Corollary 8.9. *For any t, either all points in $a^{-1}(t)$ are interior, or none of them are so, except those $\tau_0 \in a^{-1}(t)$ around which the solution $x(t;\tau_0,0)$ has local maximum (case (v) of Corollary 8.3).*

8.1. Proof of the Injectivity Theorem

We have to prove here that (when $\mathrm{int}(D_a) \neq \emptyset$) the firing map a is injective in $\mathrm{int}(D_a)$ if and only if $F(t,0) \geqslant 0$ for all $t \in \mathrm{int}(D_a)$. If the firing map a is injective and that there is $\tau_0 \in \mathrm{int}(D_a)$ such that $F(\tau_0,0) < 0$, by Lemma 8.4, $x(\tau';\tau_0,0) < 0$, for some τ' satisfying $\tau_0 < \tau' < a(\tau_0)$. Now, by the Intermediate Value Theorem, there is τ_1 such that $x(\tau_1;\tau_0,0) = 0$ and $\tau_0 < \tau_1 < a(\tau_0)$. Hence $\tau_1 \in D_a$, $\tau_1 \neq \tau_0$, $a(\tau_1) = a(\tau_0)$. It only remains to show that $\tau_1 \in \mathrm{int}(D_a)$, in order to contradict that the firing map a, is injective in the interior of D_a. Since $\tau_0 \in \mathrm{int}(D_a)$ and $F(\tau_0,0) < 0$, from Proposition 8.8, we have that the solution $x(\cdot;\tau_0,0)$ crosses ascendantly the line $x = 1$ at $(t,1)$, for some $t \geqslant a(\tau_0)$, we can say the same for the solution $x(\cdot;\tau_1,0)$ (since the solutions $x(\cdot;\tau_0,0)$ and $x(\cdot;\tau_1,0)$ match at $t = a(\tau_1) = a(\tau_0)$); thus applying again Proposition 8.8 we have that $\tau_1 \in \mathrm{int}(D_a)$.

Now, in order to prove the converse of the theorem (by contradiction) we assume that $F(\tau,0) \geqslant 0$ for all $\tau \in \mathrm{int}(D_a)$ and that the firing map is not injective in the interior of D_a. Let $t_0 \in a(\mathrm{int}(D_a))$. If the solution $x(t;t_0,1)$ attains a maximum value at some $\tau_0 \in a^{-1}(t_0)$, then, by Proposition 8.8, τ_0 is an interior point of D_a. Now by Lemma 8.5 we have that, arbitrarily close to τ_0 (and hence in $\mathrm{int}(D_a)$), there are times τ such that $F(\tau,0) < 0$, contradicting our hypothesis.

Let us consider that for all points τ in $a^{-1}(t_0)$ the solution $x(t;\tau,0)$ does not attain a maximum value at τ. Since a is not injective, there exists $t_0 \in a(\mathrm{int}(D_a))$ such that the set $a^{-1}(t_0)$ has at least two points, by the other hand, for all $t_0 \in \mathbb{R}$, the set $a^{-1}(t_0)$ is upperly bounded by t_0 and hence it must have a supremum value. This supreme value belongs to the set $a^{-1}(t_0)$, otherwise this set would have an accumulation point, τ^*. Since a point τ is in $a^{-1}(t_0)$ if and only if $\tau < t_0$ and $(\tau, 0)$ is a point in the graph Γ of the solution $x(\cdot; t_0, 1)$, there is a sequence $\{(\tau_n, 0)\}_{n=1}^{\infty}$ in Γ which converges to $(\tau^*, 0)$. Reordering, eliminating and adding points, if necessary, to the sequence $\{(\tau_n, 0)\}_{n=1}^{\infty}$, we can assume that the sequence τ_n is such that $\tau_n < \tau_{n+1}$ and, if $F(\tau_n, 0) > 0$, then $F(\tau_{n+1}, 0) \leqslant 0$ or, if $F(\tau_n, 0) \leqslant 0$, then $F(\tau_{n+1}, 0) > 0$ for all $n = 1, 2, \ldots$ Therefore, the intermediate value theorem implies that, for all $n = 1, 2, \ldots$, there is $\tau_n < \tau'_n < \tau_{n+1}$ such that

$$F(\tau'_n, 0) = 0.$$

Since the sequence τ'_n also converges to τ^* and the function $F(\cdot, 0)$ is analytic in $\mathbb{R}$, Lemma 8.1 implies $F(\tau, 0) = 0$ for all τ in $\mathbb{R}$, contradicting that $a^{-1}(t_0) \neq \emptyset$.

Let

$$\tau_2 = \max\left\{\tau \in a^{-1}(t_0)\right\}.$$

Using a similar argument we can also prove that there is

$$\tau_1 = \max\left\{\tau \in a^{-1}(t_0) : \tau < \tau_2\right\}.$$

Since we are considering the case for which the solution $x(t; \tau, 0)$ does not attain a maximum value at τ, for all points τ in $a^{-1}(t_0)$ by Corollary 8.9, τ_1 and τ_2 are interior points. We also have that $F(\tau_2, 0)$ must be greater or equal to zero, otherwise the Intermediate Value Theorem would imply the existence of an element in $a^{-1}(t_0)$ greater than τ_2. Using a similar argument, we conclude that, $x(t) = x(t; \tau_1, 0) = x(t; \tau_2, 0) > 0$ in the interval (τ_2, t_0) and we have the following alternatives for the solution $x(t)$:

(i) τ_1 or τ_2 are extremal points (local minima) of the solution $x(t)$,

(ii) neither τ_1 nor τ_2 are extremal points.

In the first case, Lemma 8.5 implies that there are $\tau \in \mathrm{int}(D_a)$ such that $F(\tau, 0) < 0$, which is contradictory. In the second case, in an interval of the form $(\tau_1 - r, \tau_1]$ we have the dichotomy shown in Proposition 8.2: either the solution $x(t)$ reaches at τ_1 a maximum or minimum value restricted to the interval $(\tau_1 - r, \tau_1]$. If the solution $x(t)$ reaches a maximum at τ_1, restricted to the interval $(\tau_1 - r, \tau_1]$, since τ_1 is not an extremal point, then $x(t) > 0$ in the interval (τ_1, τ_2) and this would imply that $x(t)$ reaches a local minimum at τ_2, which contradicts the assumption (ii). Hence $x(t)$ must reach a minimum at τ_1, restricted to the interval $(\tau_1 - r, \tau_1]$ and Lemma 8.5 implies the contradictory fact that $F(\tau, 0) < 0$ for τ arbitrarily close to τ_1 (and hence in $\mathrm{int}(D_a)$).

Thus, $F(\tau, 0) \geqslant 0$ implies the firing map injective. □

Remark 8.1. When D_a contains isolated points the condition $F(\tau, 0) \geqslant 0$, for all $\tau \in D_a$ does not necessarily imply that the firing map, a, is injective. In Fig. 5 we can see an example.

8.2. Proof of the Continuity Theorem

Some facts that will be used in the proof are interesting in their own and therefore will be first established in the form of propositions and lemmas.

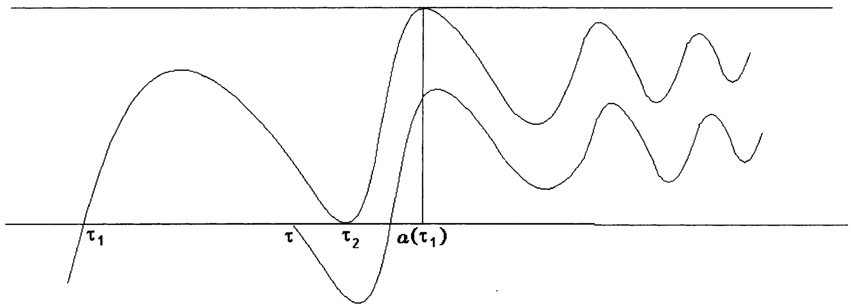

Fig. 5. $\tau_1,\ \tau_2 \in D_a$, τ_2 is an isolated point of D_a.

Lemma 8.10. *Let $J \subset I = [a,b]$ be a closed interval and suppose that for all $\tau \in J, I$ is contained in the interval of definition of the solution $x(t;\tau,0)$. Then the map from J to the space of continuous functions $C(I)$, given by*

$$\tau \mapsto x(\cdot;\tau,o),$$

is continuous with the uniform norm in C(I).

Proof. For each $\tau \in J$,

$$x(t;\tau,0) = \int_\tau^t F(s,x(s;\tau,0))\,\mathrm{d}s.$$

Let $\tau,\ \tau_0$ be fixed in J, for all t in I,

$$\begin{aligned}|x(t;\tau,0) - x(t;\tau_0,0)| &= \left|\int_\tau^t F(s,x(s;\tau,0))\,\mathrm{d}s - \int_{\tau_0}^t F(s,x(s;\tau,0))\,\mathrm{d}s\right| \\ &= \left|\int_\tau^{\tau_0} F(s,x(s;\tau,0))\,\mathrm{d}s + \int_{\tau_0}^t (F(s,x(s;\tau,0)) - F(s,x(s;\tau_0,0)))\,\mathrm{d}s\right| \\ &\leqslant \left|\int_\tau^{\tau_0} F(s,x(s;\tau,0))\,\mathrm{d}s\right| + \left|\int_a^b (F(s,x(s;\tau,0)) - F(s,x(s;\tau_0,0)))\,\mathrm{d}s\right| \\ &< \int_a^b |F(s,x(s;\tau,0)) - F(s,x(s;\tau_0,0))|\,\mathrm{d}s + M|\tau-\tau_0|,\end{aligned}$$

where $M = \max_{(s,\tau)} |F(s,x(s;\tau,0))|$.

Let $\varepsilon > 0$ and let δ_1 be such that if $|\tau - \tau_0| < \delta_1$, then

$$|F(s,x(s;\tau,0)) - F(s,x(s;\tau_0,0))| < \frac{\varepsilon}{2(b-a)}.$$

Let $\delta = \min\{\varepsilon/2M, \delta_1\}$. Hence, if $|\tau-\tau_0| < \delta$, then $M|\tau-\tau_0| < \varepsilon/2$ and

$$\int_a^b |F(s,x(s;\tau,0)) - F(s,x(s;\tau_0,0))|\,\mathrm{d}s < \frac{\varepsilon}{2},$$

therefore, for all $t \in I$,

$$|x(t;\tau,0) - x(t;\tau_0,0)| < \varepsilon \quad \text{if } |\tau - \tau_0| < \delta. \qquad \square$$

The following propositions will imply that, for the integrate and fire systems of the type that we are considering, at any point $t \in D_a$, the firing map, a, either is continuous or has jump discontinuities (i.e., at any point of discontinuity the map a is continuous from one side and there is the limit from the other side).

Proposition 8.11. *Let $\tau_0 \in D_a$.*
(i) *If there is $r > 0$ such that for $\tau_0 - r < \tau \leqslant \tau_0$, $F(\tau,0) \geqslant 0$, then a is left continuous in τ_0.*
(ii) *If there is $r > 0$ such that for $\tau_0 \leqslant \tau < \tau_0 + r$, $F(t,0) \leqslant 0$, then a is right continuous in τ_0.*

Proof. We will discuss only the first case, the other one being completely analogous. Suppose that $r > 0$ is such that the interval $[\tau_0 - r, \tau_0]$ is contained in the interval of definition of the solution $x(t;\tau_0,0)$ and such that $x(t;\tau_0,0) < 0$ for $t \in [\tau_0 - r, \tau_0)$ (cf. Lemma 8.4). Then, by uniqueness, for any given $\tau \in [\tau_0 - r, \tau_0]$, we have that $x(t;\tau,0) \geqslant x(t;\tau_0,0)$ for all $t \in [\tau, a(\tau_0)]$, hence $\tau \in D_a$ and $a(\tau) \leqslant a(\tau_0)$ for $\tau \in [\tau_0 - r, \tau_0]$.

Let $\varepsilon > 0$ and let

$$\varepsilon' = \inf_{t \in [\tau_0 - r, a(\tau_0) - \varepsilon]} \{1 - x(t;\tau_0,0)\}.$$

Applying Lemma 8.10 with $J = [\tau_0 - r, \tau_0]$ and $I = [\tau_0 - r, a(\tau_0) - \varepsilon]$, there is $\delta' > 0$ such that if $|\tau - \tau_0| < \delta'$, then

$$|x(t;\tau,0) - x(t;\tau_0,0)| < \varepsilon' \quad \text{for all } t \in [\tau_0 - r, a(\tau_0) - \varepsilon].$$

This implies that $x(t;\tau,0) < 1$ for all $t \in I$ and therefore the system does not fires during the lapse I. Now let $\delta = \min\{\delta', r\}$, then, if $\tau_0 - \delta < \tau \leqslant \tau_0$,

$$a(\tau_0) - \varepsilon < a(\tau) \leqslant a(\tau_0).$$

and the proposition is thereby proved. $\square$

Proposition 8.12. *Assume that $\tau_0 \in \text{int}(D_a)$, and the solution $x(t;\tau_0,0)$ crosses ascendantly the line $x = 1$ at some point $(t,1)$, with $t \geqslant a(\tau_0)$, then*
(i) *if there is $r > 0$ such that $F(\tau,0) \geqslant 0$, for $\tau_0 \leqslant \tau < \tau_0 + r$,*

$$\lim_{\tau \to \tau_0^+} a(\tau) = t^*,$$

(ii) *if there is $r > 0$ such that $F(\tau,0) \leqslant 0$, for $\tau_0 - r \leqslant \tau < \tau_0$,*

$$\lim_{\tau \to \tau_0^-} a(\tau) = t^*,$$

where t^* is the minimum time $t \geqslant a(\tau_0)$ such that the solution $x(t;\tau_0,0)$ crosses ascendantly the line $x = 1$ at $(t,1)$.

Remark 8.2. The minimum value t^* exists as consequence of Lemma 8.1 (cf. Proof of Injectivity Theorem).

Proof. Case (i): If t^* is the minimum time $t \geqslant a(\tau_0)$ such that the solution $x(t;\tau_0,0)$ crosses ascendantly the line $x = 1$ at $(t,1)$, there is $s > 0$ such that $F(t,1) > 0$ if $0 < |t - t^*| < s$. By Lemma 8.7 the function

$$h : (t^* - s, t^* + s) \to \mathbb{R}$$

$$h(t) = x(t^*; t, 1)$$

is a homeomorphism and $h(t^*) = 1$. Also we have that $x(t;\tau_0,0) \leqslant 1$ for all $t \in [\tau_0, t^*]$, otherwise, if there is $\tau_0 < t \leqslant t^*$ such that $x(t;\tau_0,0) > 1$, Corollary 8.6 implies that the solution $x(t;\tau_0,0)$ crosses ascendantly the line $x = 1$, at some $(t',1)$ with $\tau_0 < t' < t^*$, which is impossible since t^* is minimum.

Given $\varepsilon > 0$, let $\lambda = \min\{\varepsilon, s\}$ and ε' such that $(1 - \varepsilon', 1 + \varepsilon') \subset h(t^* - \lambda, t^* + \lambda)$ (see Fig. 6). By the continuity of the solutions with respect to the initial conditions, there is $\delta > 0$ such that if $|\tau - \tau_0| < \delta$, then

$$|x(t^*;\tau,0) - x(t^*;\tau_0,0)| < \varepsilon'.$$

Also suppose that δ is such that $x(t;\tau_0,0) > 0$ if $\tau_0 \leqslant t < \tau_0 + \delta$. Then, for $\tau \in D_a$, with $\tau_0 < \tau < \tau_0 + \delta$, by uniqueness, we have $x(t;\tau,0) < x(t;\tau_0,0) \leqslant 1$ for all $t \in [\tau, t^*]$, thus necessarily $a(\tau) > t^*$.

Since $x(t^*;\tau,0) = x(t^*; h^{-1}(x(t^*;\tau,0)),0)$ by uniqueness of the solutions we have

$$a(\tau) = h^{-1}(x(t^*;\tau,0))$$

and hence $|a(\tau) - t^*| < \varepsilon$ if $\tau_0 \leqslant \tau < \tau_0 + \delta$. This proves (i); the proof of (ii) is analogous. □

Lemma 8.13. *Let us suppose that* $\tau_0 \in \mathrm{int}(D_a)$ *and the solution* $x(t;\tau_0,0)$ *is such that there is an* $\alpha > 0$ *such that* $F(t,1) < 0$ *for* $t \in (a(\tau_0), a(\tau_0) + \alpha)$. *Then*

(i) *If there is* $r > 0$ *such that* $F(\tau, 0) \geqslant 0$, *for* $\tau_0 \leqslant \tau < \tau_0 + r$, *then*

$$\lim_{\tau \to \tau_0^+} a(\tau) \neq a(\tau_0).$$

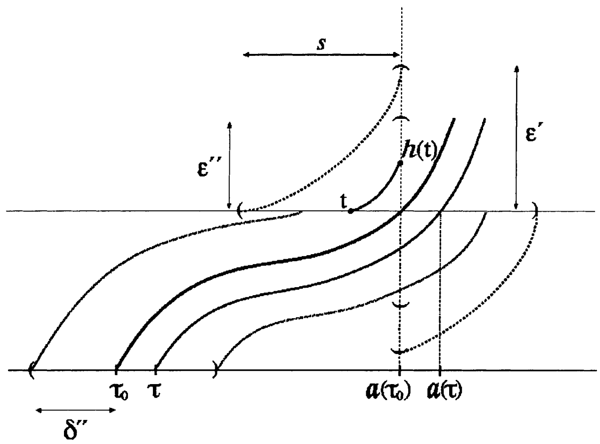

Fig. 6. The map h, defined in Lemma 8.7, and intervals involved in the Proof of proposition 8.12.

(ii) *If there is* $r > 0$ *such that* $F(\tau, 0) \leqslant 0$, *for* $\tau_0 - r \leqslant \tau < \tau_0$, *then*

$$\lim_{\tau \to \tau_0^-} a(\tau) \neq a(\tau_0).$$

Proof. In an neighborhood to the right of $a(\tau_0)$ we have three possibilities for the solution $x(t, \tau_0, 0)$, either it is constant or it has a minimum or a maximum at $a(\tau_0)$. From Lemma 8.5 we have that it must has a maximum. Therefore, there is $\varepsilon > 0$ such that $x(t; \tau_0, 0) < 1$ for all $t \in (a(\tau_0), a(\tau_0) + \varepsilon)$. Let us assume now the hypothesis of (i), the proof of (ii) is similar. The uniqueness of solutions implies that, for τ close enough and larger than τ_0, $x(t; \tau, 0) < x(t; \tau_0, 0) \leqslant 1$ for all $t \in (\tau, a(\tau_0) + \varepsilon)$. Hence, for those times τ at the right of τ_0, the firing time $a(\tau)$ cannot be in the neighborhood $(a(\tau_0), a(\tau_0) + \varepsilon)$ of $a(\tau_0)$. That is,

$$\lim_{\tau \to \tau_0^+} a(\tau) \neq a(\tau_0). \qquad \square$$

Proof of the Continuity theorem. Let $\tau_0 \in \mathrm{int}(D_a)$ be such that $F(t, 1) \geqslant 0$ for all t in a neighborhood of $a(\tau_0)$. Under this condition we assert that, for any t in such neighborhood, the crossing of the solution $x(\cdot; t, 1)$ with the line $x = 1$, at $t = 1$ is ascendant: If $F(t, 1) > 0$, it follows from continuity. If $F(t, 1) = 0$, since F is analytic, there is an neighborhood N of t such that $F(s, 1) \neq 0$ for all $s \in N$, $s \neq t$. Then, in small enough neighborhood of t, $F(s, 1) > 0$, so the solution with initial conditions $(t, 1)$ crosses ascendantly at $(t, 1)$. In particular, the solution $x(\cdot; a(\tau_0), 0)$ will cross ascendantly the line $x = 1$, at $t = 1$.

Since the solution $x(\cdot; \tau_0, 0)$ is analytic and it is not constant, by Corollary 8.3 we have the following alternatives in an neighborhood of τ_0: (i) it has a local maximum; (ii) it has a local minimum; (iii) it is strictly increasing; (iv) it is strictly decreasing. In case (i) Lemma 8.5 and Proposition 8.11 give us that a is both left and right continuous at τ_0. In case (ii) Lemma 8.5 and Proposition 8.12 give us that the left- and the right-hand side limits of a at τ_0 coincide and is $a(\tau_0)$. In case (iii) Lemma 8.5 and Proposition 8.11 give us that a is left continuous at τ_0. By the other hand, Lemma 8.5 and Proposition 8.12 guarantee that the right-hand side limit of a at τ_0 is the first ascendant crossing (for $t \geqslant a(\tau_0)$) of the solution $x(\cdot; \tau_0, 0)$ with the line $x = 1$, but, since we have proved that, under our hypothesis, the solution $x(\cdot; a(\tau_0), 0)$ will cross ascendantly the line $x = 1$, at $t = 1$, we have that $a(t) \to a(\tau_0)$ from the right. The case (ii) is analogous. Hence a is continuous at τ_0.

To prove the converse consider $\tau_0 \in \mathrm{int}(D_a)$ and assume that a is continuous at τ_0, from Lemma 8.5 we have only two options for the local behavior of the solution $x(\cdot; \tau_0, 0)$ at the right of $a(\tau_0)$: either it has a maximum or a minimum. If $x(\cdot; \tau_0, 0)$ has a maximum at $a(\tau_0)$, this lemma gives us that for r small enough, $F(\cdot, 1) < 0$ in an neighborhood of the form $(a(\tau_0), a(\tau_0) + r)$. Then Lemma 8.13 would contradict the continuity of a at τ_0. If it has a minimum the same lemma implies that $F(\cdot, 1) > 0$ in an neighborhood of the form $(a(\tau_0), a(\tau_0) + r)$, with r small enough, which proves the theorem. $\square$

8.3. Summary and discussion

It is a well-known fact that the synchronization properties of periodical forced integrate and fire oscillators are encoded by the dynamics of firing maps from the circle to itself. Furthermore, these circle dynamics determine the behavior of the forced oscillator: the existence of periodic attractors corresponds to synchronized oscillations, quasiperiodic motions corresponds to orbits that are dense in the circle (or in a Cantor set of it), meanwhile disordered behavior correlates to orbits with positive Liapunov exponents. Being the standard tool to analyze dynamical systems in the circumference, Poincare's rotation theory, and its generalizations, beautifully applies to analyze the forced oscillator dynamics. According to rotation theory, the presence of periodic (phase locked), quasiperiodic or chaotic behavior is determined by the regularity properties (continuity and injectivity) of the oscillators firing map. The difficulty to carry out this analysis is in the fact that these firing maps are generally inaccessible: even for exactly soluble linear models, the corresponding firing maps are not analytically obtainable. However, for linear systems, ingenious methods have been devised to indirectly deduce the regularity properties of the firing maps, from equations that implicitly determine them [5,6,20,21].

In this paper we have presented theorems that allow to identify the regularity properties (continuity and injectivity) of the oscillators firing map. These theorems provide a new approach to analyze and predict the qualitative type of the integrate and fire oscillator response. This new way of analyzing firing maps has two important advantages over previous methods:

1. It provides a priori conditions to determine the regularity type of the firing functions (and therefore the qualitative type of the oscillators response), that it to say, conditions that can be determined directly from the right-hand side of the differential equation of the model, without requiring any knowledge about its solutions.
2. The method also applies to analyze non-linear systems.

This last advantage is particularly important since non-linear models cover a wider range of phenomena and bring more plasticity to model neuronal behavior. The non-linear example that we have analyzed, not only has shown how direct and simple is to apply our analysis method, but have confirmed (from the mathematical point of view) that non-linear integrate and fire models express a richer phenomenology than the linear ones. It is to be expected that this mathematical breakthrough will stimulate scientists interested in forced oscillators to use and analyze, using this new methodology, non-linear integrate and fire models.

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