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ON THE DYNAMICS OF THE ONE PARAMETER FUNCTIONS $F_a(z) = z^2 + 2a\bar{z}$

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Abstract

We associate the set $K(F_a)$, to the family of functions $F_a(z) = z^2+2a\bar{z}$, where $z \in \mathbb{C}$ and $a \in \mathbb{R}$, $K(F_a)$ is the set points in $\mathbb{C}$ whose orbit under F_a is bounded. We describe the bifurcations of F_a and some of its dynamics on $K(F_a)$, focusing mainly on the connectedness of $K(F_a)$.

Introduction

The quadratic mappings $f_c(z) = z^2 + c$, $z \in \mathbb{C}$ have been studied by many authors (Douady, Hubbard, Yoccozz, et. al), the dynamics of this family of holomorphic maps is encoded by the well-known Mandelbrot set. In fact, if $J(f_c)$ denotes the Julia set for the above maps, the set of parameter values c for which $J(f_c)$ is connected defines the Mandelbrot set.

More recently J. Milnor, R.Winters [7] and others have studied the equivalent to the Mandelbrot set for the family of antiholomorphic maps defined by $g_c(z) = \bar{z}^2 + c$.

On the other hand G. Gómez and S. López de Medrano studied from the dynamical point of view a classification of families of quadratic maps (with singularities) from $\mathbb{R}^2$ to $\mathbb{R}^2$, see [3]. In their work, they ask to what extent the behaviour of the dynamics of holomorphic mappings can be extended to non-holomorphic maps. One of the families in the classification given in [3] is $F_a(z) = z^2 + 2a\bar{z}$, $a \in \mathbb{R}$, for which the authors constructed computational images of $J(F_a)$ (see Definition 2) for some values of a.

The above family $F_a(z)$ shares with the holomorphic family $f_c(z)$ the fact that the singular set of both functions is compact and that ∞ is an attractive fixed point (see Lemma (3)).The singular set of $F_a(z)$, is a circle (see §0), while the singular set of $f_c(z)$ is a point. Moreover, $F_a(z)$ is a universal unfolding ($a \in \mathbb{C}$) for the map $F_0(z) = z^2$ (see the Appendix)

In this paper we investigate the connectivity of $J(F_a)$, proving that $J(F_a)$ is connected if and only if $a \in [-1, 2]$ (see theorems (1), (2), (3)). As in the complex case, the singular set of F_a plays an important role in proving the

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above theorems. We use Whitney's theory on classification of singularities for maps of $\mathbb{R}^2$ and well known topological techniqhes used in holomorphic dynamics to prove these theorems. The proof breaks down into several cases according to the behaviour of the singularity set of F_a under iteration. The singular set $\sum_a$ behaves basically in three different ways as we iterate it:

1. If $F_a(\Sigma_a)$ remains bounded as $n \to \infty$ $(-1 \leq a \leq \frac{1}{3})$, then $J(F_a)$ is connected.
2. If some points of $F_a^n(\Sigma_a)$ go to ∞, but the point $-a$ has bounded orbit ($\frac{1}{3} < a < 2$), then $J(F_a)$ is connected.
3. If $-a \in F^n(\Sigma_a) \to \infty$ as $n \to \infty$ ($a > 2$ or $a < -1$), then $J(F_a)$ is disconnected.

At the same time we study some of the dynamics of F_a on $J(F_a)$ for $a \in [-1, 2]$. For instance, using a λ-lemma argument we prove that the stable manifold of a saddle fixed point of F_a is contained in $J(F_a)$, (see Corollaries (1) and (2)).

§ **0.** In this section we will establish some basic facts and properties of the functions F_a.

First, observe that the functions F_a are not holomorphic, and have the following properties:

i) If $r \in \mathbb{R}$, then $F_a(r) \in \mathbb{R}$.

ii) If ρ is a cube root of unity then:

$$F_a(\rho z) = \bar{\rho}(z^2 + 2a\bar{z}) = \rho^2 F_a(z),$$
$$F_a(\rho^2 z) = \rho(F_a(z)).$$

iii) $F_a(\bar{z}) = \bar{F}_a(z)$.

Writing $F_a(z)$ in real coordinates, we obtain $F_a(z) = F_a(x, y) = (x^2 - y^2 + 2ax, 2yx - 2ay)$ with derivative

$$DF_a(x, y) = \begin{pmatrix} 2x + 2a & -2y \\ 2y & 2x - 2a \end{pmatrix}.$$

Hence, the singular set of $F_a(z)$, which we will denote by $\sum_a$, is the set $\{x^2+y^2 = a^2\}$, i.e., the circle of radius a centered at $\bar{0} = (0, 0)$. This implies, in particular, that the functions $F_a(z)$ are not quasiconformal if $a \neq 0$.

For $z_0 = (x_0, y_0)$, the eigenvalues of the derivative at z_0 are $\lambda_{\pm}(z_0) = 2(x_0 \pm \sqrt{a^2 - y_0^2})$.

The fixed points of $F_a(z)$ are $\bar{0}$, $p_0 = 1 - 2a$, $p_1 = (1/2 + a, \sqrt{3a^2 + a - 1/4})$ and $p_2 = (1/2 + a, -\sqrt{3a^2 + a - 1/4})$ where p_1 and p_2 do not exist if $a \in [-1/2, 1/6]$. Due to condition (ii) above, ρp_i and $\rho^2 p_i$ are orbits of period two for $i = 0, 1, 2$.

The restriction $F_a\mid_{\mathbb{R}}$ is $F_a(r) = r^2 + 2ar$ and is topologically conjugate to the function $f_c(r) = r^2 + c$, where $c = -a^2 + a$, by the affine change of coordinates $r \mapsto r + a$.

The fixed points for $F_a\mid_{\mathbb{R}}$ are $\bar{0}$ and $p_0 = 1 - 2a$.

Also, the singular set is the point $-a$ and $F_a'(0) = 2a$; $F_a'(p_0) = 2 - 2a$ which coincides with the λ_+ eigenvalue.

For $(x_0, y_0) \in \sum_a$, $\lambda_+(x_0, y_0) = 4x_0$ and $\lambda_-(x_0, y_0) = 0$. Thus on $\sum_a$, the differential of F_a has rank one.

In [6] Whitney introduced the concepts of fold and cusp maps which we will use in the following proposition.

The theorems of Whitney on singularities (see [2] or [6]) establish that if $p \in \mathbb{R}^2$ is a fold point for a function f, then f is equivalent to $(x, y) \to (x^2, y)$ at $\bar{0}$, and if p is a cusp point for f, then f is equivalent to $(x, y) \to (x^3/3 + xy, y)$ at $\bar{0}$. Moreover, the set of functions $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ whose singular points are folds and cusps are dense in the C^∞ topology.

For every parameter value, the point a is in $\sum_a$ and we have:

PROPOSITION (1). *If $a \neq 0$, for $z \in \sum_a - \{a, \rho a, \rho^2 a\}$ there is a neighborhood of z, N_z, such that $F/_{N_z}$ is equivalent to a fold map; and for $z \in \{a, \rho a, \rho^2 a\}$, $F/_{N_z}$ is equivalent to a cusp map.*

Proof. Since Σ_a is a differentiable curve, a point $p \in \Sigma_a$ is by definition a fold point if $F_a/\sum_a$ is regular at p and $p \in \sum_a$ is a cusp point if $(F_a/\sum_a)'(p) = 0$ and $(F_a/\sum_a)''(p) \neq 0$. Parametrizing $\sum_a$ as $t \mapsto ae^{it}$, $t \in [0, 2\pi]$, we have $F_a(ae^{it}) = a^2e^{2it} + 2a^2e^{-it}$, so $\frac{d}{dt}F_a(ae^{it}) = a^2 2ie^{2it} - i2a^2e^{it}$. Hence $\frac{d}{dt}F_a(ae^{it}) = 0$ if $e^{2it} - e^{-it} = 0$, i.e., $e^{3it} = 1$, which implies $t = 0, 2\pi/3, 4\pi/3$. $F_a/\sum_a$ is regular at $\sum_a - \{a, \rho a, \rho^2 a\}$ and any point in this set is a fold point. On the other hand, $\frac{d^2}{dt^2}F_a(ae^{it}) = -2a^2(2e^{2it} + e^{-it}) \neq 0$ for $t = 0, 2\pi/3, 4\pi/3$ and so $a, \rho a, \rho^2 a$ are cusp points. This proves the proposition. ■

One has that $F_a(\sum_a)$ is a hypocycloid of three cusps (the cusps being $F_a(a)$, $F_a(\rho a)$, $F_a(\rho^2 a)$). The set $(a, \sum_a) \subset \mathbb{R}^3$ is the elliptic umbilic set of the elementary catastrophes (see [1]).

1. Writing z as $re^{i\Theta}$ we have that $|F_a(z)| = |z^2 + 2a\bar{z}| = |r^2e^{2i\Theta} + 2are^{-i\Theta}|$; for each r, this quantity has a maximum at $\Theta = 0$ and a minimum at $\Theta = \pi$ if $a > 0$, and viceversa if $a < 0$. The point $\bar{0} \in \mathbb{R}^2$ is an attractive fixed point for $F_a(z)$ if $-1/2 < a < 1/2$, since the eigenvalues of the derivative are $\lambda_\pm(0) = \pm 2a$. The fixed point $p_1 = 1 - 2a$ is expansive for $a < 1/6$ and a saddle if $1/2 > a > 1/6$.

LEMMA (1). *If $-1/2 < a < 1/2$ and $|z| < 1 - 2a$, then $|F_a^n(z)| \to 0$ as $n \to \infty$; moreover, $|F_a^n(z)| < |F_a^{n-1}(z)|$ for all $n > 0$.*

Proof. Since the function $|F_a(z)|$ has a maximum at $\Theta = 0$, then $|F_a(z)| \leq |r^2 + 2ar|$, where $z = re^{i\Theta}$. This means that $|F_a(z)|$ is bounded by the image of the point r under the map $F_a|_{\mathbb{R}}$. As we have seen in §0, the map $F_a|_{\mathbb{R}}$ is conjugate to the map $F_c(r) = r^2 + c$ with $c = -a^2 + a$; if $-1/2 < a < 1/2$ then $-3/4 < c < 1/4$. For these values of c, the map f_c has two fixed points in $\mathbb{R}$ which are $(1-\sqrt{1-4c})/2$ and $(1+\sqrt{1-4c})/2$, one attractive and the other repelling, respectively. It is known (see [5] Section 11.1) that the interval $[-((1+\sqrt{1-4c})/2), (1+\sqrt{1-4c})/2]$ is mapped inside itself under f_c, and every point inside this interval tends uniformly towards the atractive point $(1-\sqrt{1-4c})/2$. The conjugation between f_c and $F_a|\mathbb{R}$ sends $\bar{0}$ to $-a$, $(1-\sqrt{1-4c})/2$ to $\bar{0}$, $(1+\sqrt{1-4c})/2$ to $1-2a$ and $-((1+\sqrt{1-4c})/2)$ to -1, so the interval $[-1, 1-2a]$ is mapped inside itself under $F_a|_{\mathbb{R}}$ and every point on it tends towards $\bar{0}$ uniformly. This proves the lemma. ∎

LEMMA (2). *If* $-1/2 < a < 1/2$ *and* $|z| > 3$, *then* $|F_a^n(z)| \to \infty$ *as* $n \to \infty$.

Proof. We have to observe that in the proof of Lemma (1), f_c maps every point outside the interval $[-((1+\sqrt{1-4c})/2), (1+\sqrt{1-4c})/2]$ to ∞, hence $F_a|$ $\mathbb{R}$ sends every point not in the interval $[-1, 1-2a]$ to infinity, and the lemma follows. ∎

In order to prove the theorems, we need to know the behavior of the inverse image under F_a of a curve that intersects the critical values.

For that, remember that the cusp map is given by the function $g: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $g(x, y) = (x^3/3 + xy, y)$. The singular set $\sum_g$ is a parabola $(t, -t^2)$ and its image $g(\sum_g)$ is the cusp $(2t^3/3, -t^2)$ or the set $\{(x, y) : x^2 = -4/9y^3\}$. The cusp set $g(\sum_g)$ divides $\mathbb{R}^2$ into three pieces:

$$R_1 = \{(x, y) : x \leq 0 \text{ and } x^2 \geq -4/9y^3\},$$
$$R_2 = \{(x, y) : x > 0 \text{ and } x^2 \geq -4/9y^3\},$$

and

$$C = \{(x, y) : x^2 < -4/9y^3\};$$

the set C is the "interior" of the cusp set.

Definition 1. We say that a continuous curve $\Gamma: [0, 1] \to \mathbb{R}^2$ is *in good position with respect to* $g(\sum_g)$ if
(a) $\Gamma \cap g(\sum_g) = \phi$, or
(b) $\Gamma \cap g(\sum_g) \neq \phi$ and there exists $t_0, t_1 \in [0, 1] (t_0 < t_1)$, such that $\Gamma[0, t_1] \in R_1$, $\Gamma(t_1, t_2) \in C$ and $\Gamma[t_2, 1] \in R_2$.
Also, Γ is good if it is the finite union of curves as above or if $\Gamma(-t)$ is good.

PROPOSITION (2). *Let* $\Gamma: [0, 1] \to \mathbb{R}^2$ *be a simple, connected curve with* $\Gamma(t) \neq \bar{0}$ *for all* $t \in [0, 1]$, *which is good with respect to* $g(\sum_g)$. *Then* $g^{-1}(\Gamma(t))$, $t \in [0, 1]$, *is also a simple connected curve.*

Proof. First, to prove that $g^{-1}(\Gamma(t))$ is connected, consider the function $h: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $h(x, y) = (x, y, x^3/3 + xy)$ and $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ with $\pi(x, y, z) = (z, y)$. Then $\pi \circ h = g$. The image of $\mathbb{R}^2$ under h defines the "cusp surface" S, and we can consider $\pi^{-1}: \mathbb{R}^2 \to S$ and $h^{-1}|_S: S \to \mathbb{R}^2$ with $h^{-1}|_S \circ \pi^{-1} = g^{-1}$ (see Fig. 1).

By hypothesis the curve $\Gamma(t)$ is good with respect to $g(\sum_g)$, so there exist $t_0, t_1 \in [0, 1]$ with $\Gamma[0, t_1] \in R_1, \Gamma(t_1, t_2) \in C$ and $\Gamma[t_2, 1] \in R_2$. The piece of curve $\Gamma[0, t_1]$ is such that $\pi^{-1}\Gamma[0, t_1]$ is a unique curve $\alpha[0, t_1]$ on S and $\pi^{-1}\Gamma[t_2, 1]$ is also a unique curve $\beta[t_2, 1]$. The curve $\Gamma[t_1, t_2]$ is such that $\pi^{-1}\Gamma[t_1, t_2]$ consist of three curves $\gamma_1[t_{11}, t_{12}], \gamma_2[t_{21}, t_{22}], \gamma_3[t_{31}, t_{32}]$ with the property that $\alpha(t_1) = \gamma_1(t_{11}), \gamma_1(t_{12}) = \gamma_2(t_{21}), \gamma_2(t_{22}) = \gamma_3(t_{31})$ and $\gamma_3(t_{32}) = \beta(t_2)$. Hence $h^{-1}|_{S \circ h^{-1}}(\Gamma[0, 1]) = g^{-1}(\Gamma[0, 1])$ is a connected curve. Outside of C, g^{-1} acts as local diffeomorphism, so the other cases follow from this observation and the discussion above.

Observe that if Γ is differentiable, $g^{-1}\Gamma$ is not necessarily differentiable at $g^{-1}(\Gamma \cap \sum_g)$.

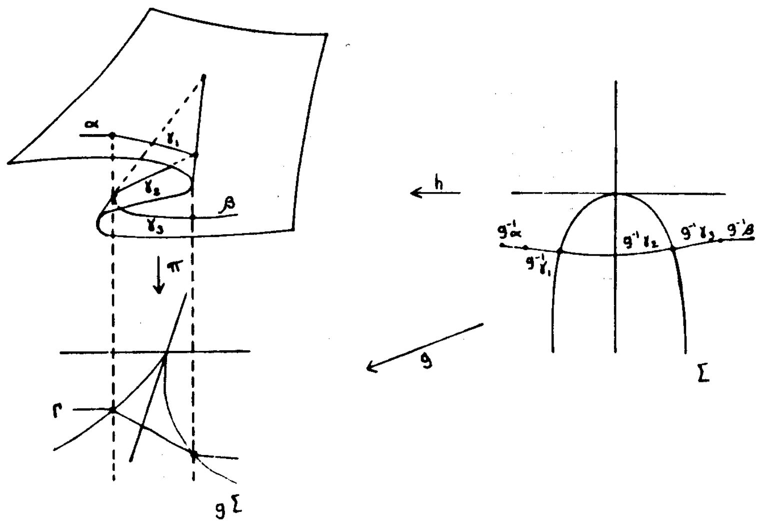

Figure 1.

Now to prove that $g^{-1}\Gamma$ is simple, suppose it is not and assume that $g^{-1}\Gamma$ has a crossing point at p_0. Then $p_0 \in \sum_g$. By hypothesis, $\Gamma \subset \mathbb{R}^2 - \{0\}$, so p_0 is a fold point. Under a fold map, any two crossing lines on a small neighborhood of p_0 project onto four lines or, if they are symmetric respect to the fold points, onto two lines, (see fig. 2a).

Thus any loop at p_0 projects onto a loop or a curve as in Figure 2b. In the first case this implies that Γ is not a simple curve, and in the second, that Γ is not good with respect to $g(\sum_g)$; in either case, we obtain a contradiction, so $g^{-1}\Gamma$ is simple. This proves the proposition.

Definition 2. For a continuous map $f: \mathbb{R}^2 \to \mathbb{R}^2$, let

$$K(f) = \{p \in \mathbb{R}^2 : |f^n(p)| \text{ is bounded for all } n\}.$$

The set $K(f)$ is formed of the points in the plane whose orbit is bounded. As in the holomorphic case we call $K(f)$ the filled-in Julia set of f.

Let $J(f) = \{p \in K(f)$: for every neighborhood V_p of p, $V_p \cap K(f)^c \neq \phi\}$ where $K(f)^c$ is the complement of $K(f)$. This set is the *Julia set of f*.

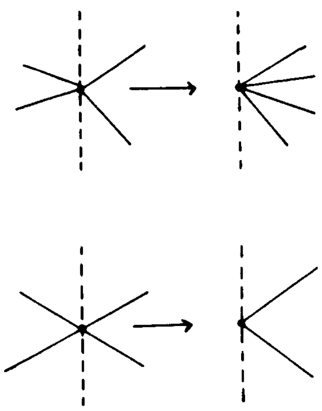

Figure 2a.

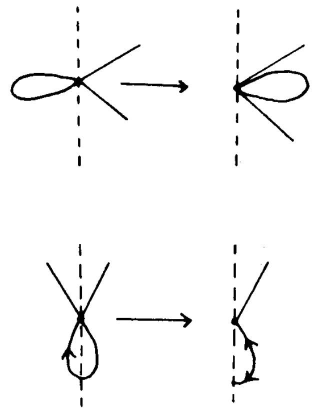

Figure 2b.

LEMMA (3). *The functions F_a extend to a C^1 map at ∞, the point ∞ is a superattractor fixed point and F_a is two to one in a neighborhood of ∞.*

Proof. To prove that the equation $z^2 + 2a\bar{z} = w$ has two solutions if $|w|$ is big enough,we have to solve the simultaneous equations $x^2 - y^2 + 2ax = w_1$ and $2xy - 2ay = w_2$ where $(w_1, w_2) = w$. The first equation represents a hyperbola whose asymptotes are the two lines that intersect at the point $(-a, 0)$ and have slope ± 1. If $w_1 \geq -a^2$ the two branches of this hyperbola intersect the real axis at $(-a - \sqrt{a^2 + w_1}, 0)$ and at $(-a + \sqrt{a^2 + w_1}, 0)$. If $w_2 \leq -a^2$, the branches of the hyperbola intersect the line $(-a, y)$ at the points $(-a, +\sqrt{a^2 - w_1})$ and at $(-a, -\sqrt{a^2 - w_1})$. The second equation represents a hyperbola with asymptotes the real axis and the line (a, y). This hyperbola intersects the imaginary axis at the point $(0, \frac{w_2}{-2a})$. Then it is easy to check that for $|w|$ big enough, the intersection of the two hyperbolas consists of two points.

To prove that F_a extends to a C^1 map, let $\gamma(z) = \frac{1}{z}$; then $\tilde{F}_a(z) = \gamma^{-1} F_a \gamma(z) = z^2\bar{z}/(\bar{z} + 2az^2)$.

It is clear that $\tilde{F}_a(0) = 0$, hence ∞ is a fixed point.

Now

$$(\tilde{F}_a)_z = \frac{2z\bar{z}(\bar{z} + 2az^2) - 4azz^2\bar{z}}{(\bar{z} + 2az^2)^2} = \frac{2z\bar{z}^2}{(\bar{z} + 2az^2)^2}$$

and

$$(\tilde{F}_a)_{\bar{z}} = \frac{z^2(\bar{z} + 2az^2) - z^2\bar{z}}{(\bar{z} + 2az^2)^2} = \frac{2az^4}{(\bar{z} + 2az^2)^2}.$$

If we divide and multiply this last equation by $\bar{z}^2$, we obtain that $\lim_{z\to 0}(\tilde{F}_a)_z = 0$ and $\lim_{z\to 0}(\tilde{F}_a)_{\bar{z}} = 0$ and so $\tilde{F}_a$ is differentiable at $\bar{0}$. Since the Jacobian of $\tilde{F}_a$ is $|(\tilde{F}_a)_z|^2 + |(\tilde{F}_a)_{\bar{z}}|^2 = 0$ then $\bar{0}$ is a superattractive fixed point.This finishes the proof.

The following basic lemma will be useful.

LEMMA (4). *Let $\{C_i\}_{i=1}^{\infty}$ a family of compact sets in $\mathbb{R}^2$ such that $C_{i+1} \subset C_i$ for all i, and such that each C_i is connected. Then $\bigcap_{i=1} C_i$ is connected.*

Let $B(\infty)$ be the basin of attraction of ∞.

THEOREM (1). *The set $K(F_a)$ is connected if $-1/2 < a < 1/2$.*

Proof. The proof will be divided in four cases. (according to the dynamics of F_a on $K(F_a)$).

Case 1: $0 < a < 1/3$:

For this case we have that $\sum_a \cap \{p_0 = 1 - 2a\}$ is empty since $a < 1 - 2a$. Let Γ_0 denote the circle of radius r about $\bar{0}$, where $r \geq 1$. Then $\Gamma_0 \subset B(\infty)$; by Lemma (1), Γ_0 contains both the attracting fixed point $\bar{0}$ and the singular set $\sum_a$ in its interior. The preimage Γ_1 of Γ_0 under F_a is a simple closed curve which is contained in the interior of Γ_0. It is mapped in a two to one fashion onto Γ_0 (by Lemma (3)). The fact that Γ_1 is a simple closed curve follows from the fact that both $\sum_a$ and its image lie inside Γ_1 (in fact $F_a^n \sum_a \to 0$ as $n \to \infty$, from Lemma (1)). The curves Γ_0 and Γ_1 bound an annular region A_1. By the same argument, there exists a curve Γ_2 which is mapped in a two to one fashion onto Γ_1. Moreover F_a maps the annular region A_2 between Γ_2 and Γ_1 onto A_1, again in a two to one fashion.

Continuing in this way we obtain a family of simple closed curves $\{\Gamma_i\}$ and annular regions A_i between them. The area of the A_i must converge to zero. Each of the curves Γ_i contains in its interior the disc of radius $1 - 2a$, by Lemma (1). This implies that the curves $\{\Gamma_i\}$ converge to a connected, closed curve denoted by Γ_∞.

Noting that $K(F_a) = \bigcap_{n\geq 0} \overline{\text{int}\Gamma_n}$ then by Lemma (4), we obtain Case 1.

Case 2: $a = 1/3$.

In this case the critical set $\sum_{1/3}$ intersects the fixed point $p_0 = 1/3$. Since $|F_a(z)| = |r^2e^{2i\Theta} + 2are^{-i\Theta}|$ has a maximum at $\Theta = 0$, for each r; then $|F_a(z)| \leq$

$|r^2 + 2ar|$. This implies that $|F_a(z)| \leq 1/3$ if $z \in \sum_{1/3}$, equality holding if z consist of the cusp points $p_0, \rho p_0, \rho^2 p_0$. If $w \in \sum_{1/3} - \{p_0, \rho p_0, \rho^2 p_0\}$ then, from the inequality above and Lemma (1), we see that $F_a^n w \to 0$ as $n \to \infty$.

As in case 1, let Γ_0 be any circle centered at $\bar{0}$ with $\Gamma_0 \in B(\infty)$; then the curve Γ_0 contains on its interior $\sum_{1/3}$ and $F_{1/3}(\sum_{1/3})$. So we can consider $\Gamma_1 = F_{1/3}^{-1}\Gamma_0$, which is a simple closed curve. Γ_1 is mapped on a two to one fashion onto Γ_0 (by Lemma (3)).

We can proceed in this way, obtaining, for each natural number i, the simple closed curve Γ_i, as in case 1, such that $\Gamma_{i+1} \subset \text{int}\Gamma_i$ and Γ_i contains in its interior $\sum_{1/3}$ and $F_{1/3}(\sum_{1/3})$. Hence $K(F_{1/3}) = \bigcap_{i \geq 0} \overline{\text{int}\Gamma_i}$ is connected by Lemma (4).

Case 3: $1/3 < a < 1/2$.

In this case the saddle fixed point p_0 is such that $|p_0| < a$. The unstable manifold of p_0, $W^u(p_0)$, is the set $(0, \infty)$ (see the proof of Lemma (1)), so $F_a^n \Sigma_a \to \infty$ as $n \to \infty$, and the point $-a \in \sum_a$ is mapped to $q = (-a^2, 0)$ with $|q| < |1 - 2a|$. Hence by Lemma (1), $F_a^n(q) \to 0$ as $n \to \infty$.

Let Γ_0 be a circle with centre at $\bar{0}$ and such that $\Gamma_0 \subset B(\infty)$. Then we can choose Γ_0 such that it intersects a neighborhood of the cusps of $F_a(\sum_a)$ in good position. Hence $F_a^{-1}(\Gamma_0) = \Gamma_1$ is a simple closed curve (by Proposition 2), which is also good with respect to the cusps of $F_a(\sum_a)$. Proceeding in this way, we obtain a family $\{\Gamma_i\}$ of simple closed curves with $\Gamma_{i+1} \subset \text{int}\Gamma_i$. Hence, $K(F_a) = \bigcap_{i \geq 0} \overline{\text{int}\Gamma_i}$ is connected by Lemma (4).

Case 4: $-1/2 < a < 0$.

In this case $\bar{0}$ is also an attractive fixed point and p_0 is now an expansive fixed point with $|p_0| > a$.

Let Γ_0 be a circle with center $\bar{0}$, such that $\Gamma_0 \subset B(\infty)$. By Lemma (1), any point $w \in \sum_a$ tends to the origin, so $F_a(\sum_a) \subset \text{int}\Gamma_0$. Now we can proceed as in Cases 1 or 2. This finishes the proof of the theorem. ■

We can be more specific about the dynamics of the functions F_a with $-1/2 < a < 1/2$.

For instance, if we consider the point p_0, with $-1/2 < a < 1/6$, we find that p_0 is an expanding fixed point which bifurcates into a saddle when $1/6 < a < 1/2$. By the stable manifold theorem, p_0 has a stable manifold $W_a^s(p_0)$ and an unstable manifold $W_a^u(p_0)$, which, as we have seen, is $(0, \infty)$.

Since ρp_0 and $\rho^2 p_0$ is a period two orbit, by property (ii) in §0, it is of sadle type when $1/6 < a < 1/2$. The unstable manifold of ρp_0 under F_a^2 is $\rho W_a^u(p_0) = \rho(0, \infty)$, and the unstable manifold of $\rho^2 p_0$ under F_a^2 is $\rho^2 W_a^w(p_0) = \rho^2(0, \infty)$.

For a point $q_i \in F_a^{-1}(p_0)$, $i = 1, 2$, one can take the component of the set $F_a^{-1}(W_a^u(p_0))$ or of $F_a^{-1}(W_a^s(p_0))$ that intersects q_i and call it $W_a^u(q_i)$, $(W_a^s(q_i)$ resp.) By the position of $W_a^u(p_0)$, $(W_a^s(p_0)$ resp.) with respect to $F_a \sum_a$, we can see that $W_a^s(q_i)$, $(W_a^s(q_i)$ resp.) are one dimensional manifolds. This is also true for $\rho p_i \in F_a^{-1}(\rho^i p_0)$, $i = 1, 2$.

Let w_i be in the backward orbit of p_0, ρp_0 or $\rho^2 p_0$.

COROLLARY (1). *The sets $W_a^s(w_i)$ are contained in $J(F_a)$ for $1/6 < a < 1/2$.*

Proof. It is enough to prove that $W^s(p_0) \subset J(F_a)$. First observe that $p_0 = 1 - 2a \in J(F_a)$. Now since p_0 is a saddle fixed point, the set $W^s(p_0)$ has a tubular neighborhood T (λ-lemma) foliated by small intervals transversal to $W^s(p_0)$ and invariant under F_a. Since every point $p \in W^s(p_0)$ tends to p_0 and $W^u(p_0) = (0, \infty)$, $W^s(p_0)$ divides T in two parts: one, of those points in T that tend to ∞; the other one, of those that tend to $\bar{0}$. This implies that $W^s(p_0) \subset J(F_a)$. ■

This corollary also implies that $F_a|J(F_a)$ is not topologically transitive.

Since the stable manifolds are differentiable curves, the boundary of $K(F_a)$ contains these curves, which are the bays that one observes in the computer graphics of $K(F_a)$ (see also [3]). There is also a "filament" on the middle point of each bay which is not in $K(F_a)$ and which corresponds to the part of $W^u(w_i)$ that tends to ∞.

If $1/6 < a$, two new repelling fixed points appear: p_1,p_2 (see Ref §0). One can see that p_1 y $p_2 \in J(F_a)$ by checking that the points $p_\epsilon = p_1 + (\epsilon, 0)$ tends to ∞ for all $\epsilon > 0$. Also, four new repelling period two points appear: ρp_1, $\rho^2 p_1$, ρp_2, $\rho^2 p_2$. We have:

COROLLARY (2). *The boundaries of $W_a^s(p_0)$, $W_a^s(\rho p_0)$, $W_a^s(\rho^2 p_0)$ are the sets $\{p_1, p_2\}$, $\{\rho p_1, \rho p_2\}$, $\{\rho^2 p_1, \rho^2 p_2\}$, respectively.*

Proof. By Corollary (1), $W_a^s(p_0)$ is in $J(F_a)$, then the boundary of $W_a^s(p_0)$ either consists of two fixed points or of an orbit of period two. In the first case the fixed points must be p_1 and p_2; in the second case one can check that the orbits of period two are $\{\rho p_0, \rho^2 p_0\}$, $\{\rho p_1, \rho^2 p_1\}$ and $\{\rho p_2, \rho^2 p_2\}$. So the boundary of $W_a^s(p_0)$ must be p_1 and p_2. Applying ρ and ρ^2, we prove the corollary. ■

2. In this section we will study the case when $1/2 \leq a \leq 2$; for these parameter values, the restriction $F_a|_{\mathbb{R}}$ is conjugate to $x \mapsto x^2 + c$ with c running from $+1/4$ to -2. Thus, $K(F_a|_{\mathbb{R}})$ is the interval $[-2a, 0]$, and if $w \in \mathbb{R} - [2a, 0]$ then $F_a^n(w) \to \infty$ as $n \to \infty$.

By property (ii) in §0, the sets $[-2a, 0]$, $\rho[-2a, 0]$, $\rho^2[-2a, 0]$ are in $K(F_a)$, so we define the set

$$T = \{\cup_{n=0}^{\infty} F_a^{-n}(\rho^i[-2a, 0]) : i = 0, 1, 2\}.$$

The set T has the property that $T \subset K(F_a)$. Notice that the dynamics of F_a on $[-2a, 0]$ as a moves towards -2, has a cascade of period doubling bifurcations.

The set T (see fig.3) is an infinite tree with all of its vertices of degree three, as can be seen from the fact that if N_0 is a small enough neighborhood of $\bar{0}$, then $N_0 \cap F_a(\sum_a) = \phi$, so F_a^{-1} maps N_0 diffeomorphically onto a neighborhood of any

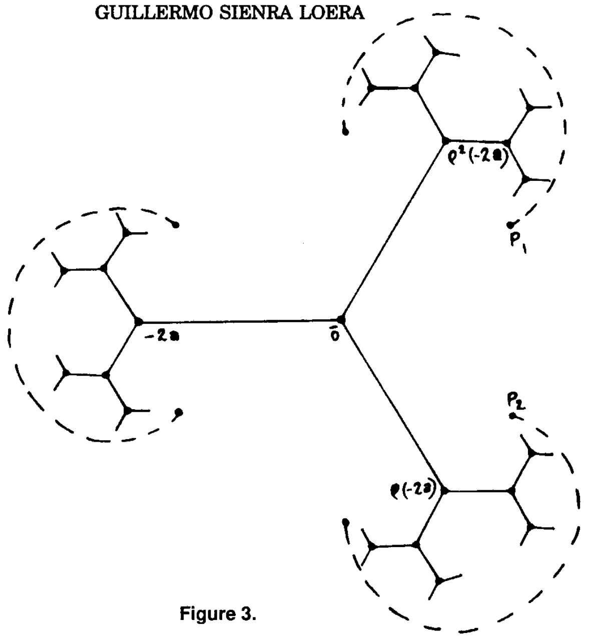

Figure 3.

of the points $\rho^i(-2a)$ since $F_a(\rho^i(-2a)) = 0$, $i = 0, 1, 2$. The same arguments apply at each point of $F_a^{-n}(\bar{0})$.

We can also observe that the set $(\infty, -2) \cup (0, \infty) = W_0$ together with ρW_0 and $\rho^2 W_0$ are points that go to ∞ with n. We can consider at each vertex of T sets $F_a^{-n}W_0, F_a^{-n}\rho W_0$, $F_a^{-n}\rho^2 W_0$ and define the set $W = \bigcap_{n=0}^{\infty}(F_a^{-n}W_0 \cup F_a^{-n}\rho W_0 \cup F_a^{-n}\rho^2 W_0)$. This set has the property that $F_a^{-1}W = W$ and for all $w \in W$, $F_a^n(w)$ tends to ∞ as n does.

THEOREM (2). *For $1/2 \leq a \leq 2$, $K(F_a)$ is connected and $K(F_a) - \bigcup_{n=0}^{\infty} F_a^{-n}(0)$ is disconnected.*

Proof. By Lemma (3), ∞ is an attractive fixed point for F_a. Also the cusp points of $\sum_a$: a, ρa, $\rho^2 a$, tend to ∞. However, the points $-a$, $-\rho a$, $-\rho^2 a$, which are in $\sum_a$, remain bounded. Let Γ_0 be a circle with center $\bar{0}$ contained in the region of attraction of ∞. For the family $\Gamma_n = F_a^{-n}\Gamma_0$ there exists a positive number N such that $\Gamma_N \cap F_a \sum_a \neq \phi$ and we can choose Γ_0 in such way that Γ_N is good with respect to $F_a(\sum_a)$.

Then, as in the proof of Theorem (1), Γ_k is a simple closed curve for $k = 0, 1, 2, \ldots$, so Γ_∞ is a connected set, implying that $K(F_a) = \bigcap_{n=0} \overline{\text{int}\Gamma_i}$ is a connected set by Lemma (4).

Due to the existence of the set W mentioned before Theorem (2), the curve Γ_∞ has crossing points at each point of the set $\bigcup_{n=0}^{\infty} F_a^{-n}(0)$, which in turn implies that $K(F_a) - \bigcup_{n=0}^{\infty} F_a^{-n}(0)$ is disconnected as we claimed. This proves the theorem. ■

Computer experiments shows that $K(F_a)$ must coincide with the set T, but we have not been able to prove it.

The dynamics of F_a undergoes several bifurcations on the set T. For instance when a is in the interval $(1/2, 2/3)$, F_a fixes $\bar{0}$ and $p_0 = 1/2a$. The point $\bar{0}$ becomes an expansive fixed point while p_0 becomes a saddle point, and $W_a^s(p_0) = (-2a, 0)$. The points ρp_0 and $\rho^2 p_0$ are period two saddle points with $W^s(\rho p) = \rho(-2a, 0) \cup \rho^2(-2a, 0)$. When $a > 3/2$, the fixed saddle point bifurcates into a period two saddle orbit, which in turn bifurcates into a period four saddle orbit and so on.

3. In this section we will deal with the remaining cases. Specifically we have:

THEOREM (3). *If $-1 < a < -1/2$, then $K(F_a)$ is a connected set.*

Proof. In this case $F_a|_{\mathbb{R}}$ is $f_a(r) = r^2+2ar$, which is conjugate to $f_c: x \mapsto x^2+c$ with c between $-3/4$ and -2, and there is a very rich dynamics as c tends to -2. However, $K(f_c) = [-(\frac{1+\sqrt{1-4c}}{2} - c)^{1/2}, \frac{1+\sqrt{1-4c}}{2}]$.

Using the affine transformation between f_a and f_c we get that $K(F_a) = [-1, 1-2a]$. Since for each r the maximum and minimum of $|F_a(z)|$ are achieved when $z \in \mathbb{R}$, $F_a^{-n}(\sum_a)$ remains bounded.

Since ∞ is an attractive fixed point by Lemma (3), and $(\infty, -1) \cup (1-2a, \infty)$ is contained in $B(\infty)$, there exists a simple closed curve Γ_0 contained in $B(\infty)$ with $\Gamma_0 \cap \mathbb{R} = \{-1\} \cup \{1-2a\} \in B(\infty)$. For all n we have $F_a^{-n}\Gamma_0 \supset \text{int}F_a^{n+1}\Gamma_0$ and for all n, $F_a^{-n}\Gamma_0$ contains $F_a^m(\sum_a)$ in its interior (for all m), so $F_a^{-n}\Gamma_0$ is a simple closed curve for all n. The limit curve $\lim_{n\to\infty} F_a^{-n}\Gamma_0$ is a connected curve, and $\bigcap_{n=0}^{\infty} \overline{\text{int}F_a^{-n}\Gamma_0}$ is an F_a- invariant set which agrees with $K(F_a)$. Thus $K(F_a)$ is a connected set by Lemma (4). This proves the theorem. ■

Finally,

THEOREM (4). *If $a > 2$ or $a < -1$, then $K(F_a)$ is a disconnected set.*

Proof. The restriction of F_a to the reals is conjugate to $x \mapsto x^2 + c$; when $a \notin [-1, 2]$, then $c < -2$, and the critical point $-a$ of $F_a|_{\mathbb{R}}$, tends to infinity.

Now since $F_a(-a) = -a^2 \in F_a(\sum_a)$, let γ_0 be a small piece of a circle tangent to $F_a(\sum_a)$ at $-a^2$ (see Figure 4).

On the other hand, ∞ is an attractive point of F_a by Lemma (3), so there exists a positive number M such that if $w \in \mathbb{R}^2$ with $|w| > M$, then $F_a^{-n}(w) \to \infty$ as $n \to \infty$. As $F_a^{-n}(-a^2)$ tends to ∞, let N be a positive integer number such that $|F_a^N(-a^2)| > M$; then $F_a^N(\gamma_0)$ is a piece of a curve that can be completed to a simple closed curve Γ_0 such that for all $w \in \Gamma_0$, $|w| > M$.

The curve Γ_0 and its exterior tends to ∞ as n does, and Γ_0 contains $\sum_a$ and $F_a(\sum_a)$ in its interior. Let $\Gamma_1 = F_a^{-1}\Gamma_0$, $\Gamma_2 = F_a^{-2}\Gamma_0$, and so on. Now $F_a^N\Gamma_0$ is, by construction, tangent to $F_a(\sum_a)$, at $-a^2$. So $F_a^{-N-1}\Gamma_0$ is a curve with a crossing point at $-a$ since this is a fold point.

Then, if D is a disc contained in the interior of Γ_0, it happens that the set $F_a^{-N-1}D$ has at least two components. Since $K(F_a) = \bigcap_{n=0}^{\infty} \overline{\text{int}F_a^{-n}\Gamma_0}$, $K(F_a)$ is disconnected. This proves the theorem. ■

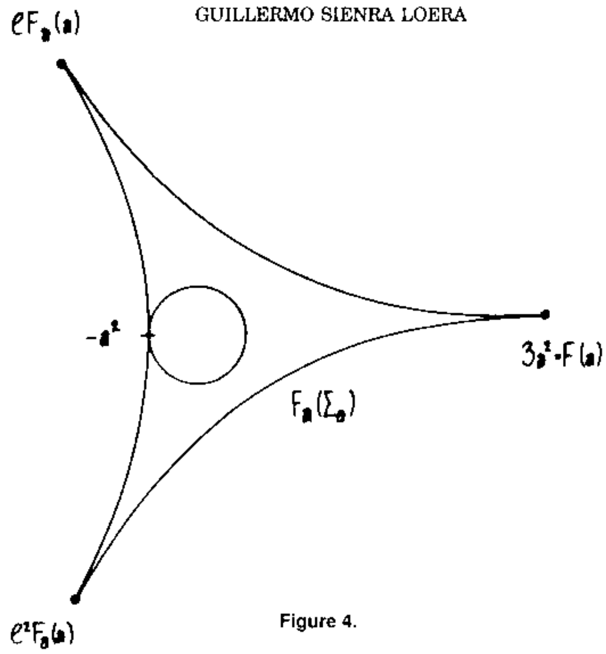

Figure 4.

From the proof of the theorem, we conclude that the set $\sum_a$ behaves in basically three different ways as we iterate it:

1) $F_a^n(\sum_a)$ remains bounded as $n \to \infty (-1 < a < 1/3)$.

2) Some points of $F_a^n(\sum_a)$ go to ∞, but $-a$ has bounded orbit $(1/3 < a < 2)$.

3) $-a \in F_a^n(\sum_a) \to \infty$ as $n \to \infty (a > 2 \; or \; a < -1)$.

Theorems (1), (2), and (3) imply that $K(F_a)$ is connected in cases (1) and (2) and disconnected in case (3).

Appendix

As we have mentioned in the Introduction, the family of maps $F_a(z) = z^2 + 2a\bar{z}$ is a universal unfolding of the map $F_0(z) = z^2$. To see that, let us take a universal unfolding for $F_0(z)$ given in [4], which is

$$G(a_1, a_2, x, y) = (a_1, a_2, x^2 - y^2 + a_1 x + a_2 y, 2xy).$$

Now, as proved in [3], the functions $g(x, y) = (x^2 - y^2 + a_1 x + a_2 y, 2xy)$ and $f(x, y) = (x^2 - y^2 + a_1 x - a_2 y, 2xy + a_2 x - a_1 y)$ satisfy $f \circ A = B \circ g$ where A and B are the affine maps given by $A(x, y) = (2x + a_1/2, 2y - a_2/2)$ and $B(x, y) = (4x + 3/4(a_1^2 - a_2^2), 4y + a_1 a_2/2)$. This implies that the unfolding G and $F(a_1, a_2, x, y) = (a_1, a_2, x^2 - y^2 + a_1 x + a_2 y, 2xy + a_2 x - a_1 y)$ are such that $G \circ \phi = \psi \circ F$, where $\phi = id \times A$ and $\psi = id \times B$ are unfoldings of the identity, hence F and G are isomorphic. Since the map F is an unfolding, then it is a universal unfolding. The function $f(x, y)$ is $F_a(z) = z^2 + 2a\bar{z}$ with $2a = a_1 + ia_2$.

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