CIRCLE MAPS AND THE DEVIL’S STAIRCASE IN A
PERIODICALLY PERTURBED OREGONATOR

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Markman and Bar-Eli has studied a periodically forced Oregonator numerically and found
a parameter range with the following properties: (1) Only periodic solutions are found in
frequency-locked steps, each with a certain pattern of large and small oscillations (2) Between
any two steps there is a step with the period being the sum of the two periods and the con-
catenation of the two patterns (3) Certain scaling properties as the period tends to infinity.
We show that such behavior occurs if the dynamics of the system is governed by a family of
diffeomorphisms of a circle with a Devil’s staircase. Using invariant manifold theory we argue
that an invariant circle must indeed exist when, as in the present case, the unforced system is
close to a saddle-loop bifurcation. Generalizations of the results are briefly discussed.

1. Introduction

Recently, Markman and Bar-Eli [1994a, 1994b] have
studied a periodically forced Oregonator numerically. In dimensionless variables the equations are
\[
\begin{align*}
\dot{x} &= a(x + y - xy) - bx^2 - k_0(t)x, \quad (1a) \\
\dot{y} &= -y + fz - xy + k_0(t)(y_0 - y), \quad (1b) \\
\dot{z} &= c(x - z) - k_0(t)z, \quad (1c)
\end{align*}
\]
and the forcing function \( k_0(t) \) is given by
\[
k_0(t) = k_0 \left( 1 + \varepsilon \sin \frac{2\pi}{T_{in}} t \right). \quad (2)
\]

Depending on system parameters, a rich variety of dynamics can occur. Here we consider what
in [Markman & Bar-Eli, 1994a] is denoted case 2 and in [Markman & Bar-Eli, 1994b] is case 3, where
parameters are fixed as shown in Table 1, and \( k_0, T_{in}, \varepsilon \) may vary as indicated. Note that only \( |\varepsilon| < 1 \)
makes sense physically but the model is well-defined mathematically for all values of \( \varepsilon \). We consider this
extended range of the parameter to get a more coherent picture of the dynamics, and, for simplicity,
we mainly study variations with \( \varepsilon \) here.

Markman and Bar-Eli make the following three basic observations:

1. Except for a very small parameter range with quasi-periodic solutions, only periodic solutions are found. The period \( T_{out} \) is a multiple of the forcing period, making the reduced period \( P = T_{out}/T_{in} \) an integer. The integer values of \( P \) occur as frequency-locked steps when regarded as a function of one of the parameters, \( \varepsilon \) or \( T_{in} \). Although the period may become arbitrarily
Table 1. Parameter values and ranges.

<table>
<thead>
<tr>
<th>Main Numerical Example</th>
<th>Ranges</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$b$</td>
</tr>
<tr>
<td>550</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Fig. 1. Plot of $x$ versus $t$ for $\varepsilon = 0$, $T_{in} = 1$, showing a periodic solution with reduced period $P = 7$ and pattern $I = LSLS^2$.

long, no bifurcations into chaos or quasi-periodic behavior seem to occur.

The solutions consist of patterns of small and large oscillations, where the large oscillations have amplitudes several thousand times the small ones. See Fig. 1. Every solution may be characterized by a finite string $I$ of $L$’s and $S$’s, where the symbols denote a large and a small maximum respectively.

The next basic observations are

2. The period-adding property: If the step at $\varepsilon = \varepsilon_1$ has a periodic solution with reduced period $P_1$ and pattern $I_1$, and the step at $\varepsilon = \varepsilon_2$ has a periodic solution with reduced period $P_2$ and pattern $I_2$, there exists a value $\varepsilon_3$ between $\varepsilon_1$ and $\varepsilon_2$ where the step has reduced period $P_3 = P_1 + P_2$ and the combined (concatenated) pattern, $I_3 = I_1I_2$. Exactly the same holds for $\varepsilon$ fixed and variation of $T_{in}$.

3. The scaling property: Given two simple patterns $I, J$. The parameter interval where the solution is locked to a solution with pattern $IJ^n, n = 1, 2, \ldots$ is denoted $A_n$ and the corresponding period is $P_n$. It is observed that $R_n = |A_n| \to 0$ and $P_n \to \infty$ as $n \to \infty$ under scalings of the form

$$P_n R_n^a = b,$$

where the exponent $a$ takes values in the range from 0.26 to 0.44.

Furthermore, one endpoint of $A_n$ converges to an endpoint $p_I$ of the parameter interval with pattern $I$. If $\Delta_n$ denotes the distance from the endpoint of $A_n$ to $p_I$, scalings of the form

$$P_n \Delta_n^c = d$$

are observed where the exponent $c$ takes values in the range from 0.51 to 0.63.

The purpose of the present paper is to give a simple and coherent explanation of these observations. We claim that this behavior is expected to occur if the dynamics of the stroboscopic map of the system is governed by a family of diffeomorphisms of a circle with a Devil’s staircase. In the unforced limit, the system is close to a saddle-node bifurcation on a limit cycle (saddle-loop bifurcation). On the basis of invariant manifold theory, we show that the stroboscopic map is indeed expected to have an invariant attracting circle for small forcings, and hence that the observed dynamics can be completely understood in this framework.

2. Reduction to a Circle Diffeomorphism

In the autonomous case, $\varepsilon = 0$, the system (1) has a saddle-loop bifurcation [Kaas-Petersen & Scott, 1988; Bar-Eli & Brøns, 1990] at $k_0 = k_0(A) \approx 0.47360$. For convenience, we introduce a new parameter $\mu = k_0 - k_0(A)$. For $\mu$ close to zero, the dynamics is depicted qualitatively in Fig. 2. For $\mu < 0$, the system has a limit cycle. At $\mu = 0$, a critical point is born on the limit cycle in a saddle-node bifurcation, destroying the limit cycle for $\mu > 0$. Note, that through the bifurcation the system has a smooth invariant attracting closed curve $\Gamma_\mu$ which for $\mu \geq 0$ consists of four orbits. In our main example, $\mu = 0.0014$, and the dynamics of Fig. 2(c) applies in the unforced case.
For \( \varepsilon > 0 \) the system is conveniently studied using the stroboscopic map or Poincaré map \( \Pi \) which takes a point \( x = (x(t), y(t), z(t)) \) on a solution curve and maps it to the point reached after one period of the forcing, \( \Pi(x) = (x(t + T_{in}), y(t + T_{in}), z(t + T_{in})) \). Thus, \( \Pi \) is a diffeomorphism: \( \mathbb{R}^3 \rightarrow \mathbb{R}^3 \). Also in the limit \( \varepsilon = 0 \) the stroboscopic map is well-defined. Hence, for sufficiently small \( \varepsilon \), the stroboscopic map is a perturbation of the unperturbed map,

\[
\Pi(x; \mu, \varepsilon, T_{in}) = \Pi_0(x; \mu, T_{in}) + \varepsilon \Pi(x; \mu, \varepsilon, T_{in}).
\]

The invariant manifolds \( \Gamma_\mu \) for the unperturbed flow are also invariant manifolds for \( \Pi_0 \). The question of persistence of invariant manifolds under perturbation of the equations is a central subject in dynamics. [Hale, 1969, Theorem 2.2, p. 232] provides a very general result in this direction. In the present setting, the theorem states that under certain conditions to be described below, the map \( \Pi \) has a smooth invariant attracting closed curve when \( \varepsilon \) is sufficiently small. Furthermore, the invariant manifold tends to the unperturbed invariant manifold as \( \varepsilon \to 0 \).

Obviously, the differential equations must fulfill certain smoothness conditions. While these are easily verified, a further technical condition is less trivial. The proof of the invariant manifold theorem includes a transformation of coordinates to a polar form, where the angular variable measures distance along the unperturbed invariant closed curve, and the radial variable (here of dimension 2) measures normal deviation from the unperturbed invariant manifold. Loosely speaking, the theorem requires that the rate of attraction in the radial direction must be larger than the Lipschitz constant of the angular parametrization. When the unperturbed invariant manifold is a limit cycle the rate condition is always fulfilled, since the appropriate Lipschitz constant is zero. However, if the invariant closed curve contains critical points it can be shown by rather simple examples [Hale, 1969, p. 238] that a violation of the rate condition can create cusps on the invariant closed curve, which then loses smoothness.

For the Oregonator, we have no means of checking this rate condition analytically. However, numerical simulations of the system show that the periodic solutions are rapidly attracting and hence that the radial motion is strongly damped, indicating that the theorem is applicable. Thus, we claim that the asymptotic dynamics of Eq. (1) as \( t \to \infty \) can be described by the restriction \( g \) of the stroboscopic map \( \Pi \) to an invariant smooth, simple, closed curve when \( \varepsilon \) is sufficiently small. We will assume that this parameter range includes the values we consider here.

Topologically, we may assume that the closed invariant curve is the circle \( S^1 \). It is worth stressing that \( g \) must be a diffeomorphism of the circle since it is the restriction of the diffeomorphism \( \Pi \). This makes the dynamics much simpler than in families of non-invertible circle maps, as studied by many authors, e.g., Hockett and Holmes [1988] and Zeng and Glass [1989].

We now proceed to describe the properties of the circle diffeomorphism which are required to account for the basic observations.

### 3. Dynamics of the Circle Diffeomorphism

#### 3.1. Basic properties

Let the invariant circle be parametrized by an angular variable \( \theta \in [0, 1] \). It follows from the expansion (5) of the stroboscopic map that the circle map \( g \) has a similar expansion

\[
g(\theta; \mu, \varepsilon, T_{in}) = g_0(\theta; \mu, T_{in}) + \varepsilon \mathcal{F}(\theta; \mu, T_{in} \varepsilon, T_{in}).
\]

For \( \varepsilon = 0 \) the dynamics can be found directly from Fig. 2 with the understanding that the critical points are now fixed points. In particular, \( g_0 \) has a saddle-node bifurcation where two fixed points are created at \( \mu = 0 \).

Let us first fix a value of \( \mu > 0 \) and let \( \varepsilon \geq 0 \) vary. For \( \varepsilon = 0 \) a stable and an unstable fixed point exist. Returning to continuous time these correspond to periodic solutions with \( P = 1 \). As shown in Table 2 the stable fixed point, which is observed in the simulations, has the pattern \( S \).
Table 2. Patterns, reduced periods, and rotation numbers for $\mu = 0.0014$ and $T_{in} = 1$.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>Pattern</th>
<th>$2L + S$</th>
<th>$P$</th>
<th>Exact</th>
<th>Approx.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.007</td>
<td>$LS^{20}$</td>
<td>22</td>
<td>23</td>
<td>1/23</td>
<td>0.043478</td>
</tr>
<tr>
<td>0.105</td>
<td>$LS^{16}$</td>
<td>18</td>
<td>19</td>
<td>1/19</td>
<td>0.052632</td>
</tr>
<tr>
<td>0.107</td>
<td>$LS^{15}$</td>
<td>17</td>
<td>18</td>
<td>1/18</td>
<td>0.055556</td>
</tr>
<tr>
<td>0.115</td>
<td>$LS^{12}$</td>
<td>14</td>
<td>15</td>
<td>1/15</td>
<td>0.066667</td>
</tr>
<tr>
<td>0.118</td>
<td>$LS^{11}$</td>
<td>13</td>
<td>14</td>
<td>1/14</td>
<td>0.071429</td>
</tr>
<tr>
<td>0.124</td>
<td>$LS^{10}$</td>
<td>12</td>
<td>13</td>
<td>1/13</td>
<td>0.076923</td>
</tr>
<tr>
<td>0.130</td>
<td>$LS^9$</td>
<td>11</td>
<td>12</td>
<td>1/12</td>
<td>0.083333</td>
</tr>
<tr>
<td>0.136</td>
<td>$LS^8$</td>
<td>10</td>
<td>11</td>
<td>1/11</td>
<td>0.090909</td>
</tr>
<tr>
<td>0.145</td>
<td>$LS^7$</td>
<td>9</td>
<td>10</td>
<td>1/10</td>
<td>0.100000</td>
</tr>
<tr>
<td>0.160</td>
<td>$LS^6$</td>
<td>8</td>
<td>9</td>
<td>1/9</td>
<td>0.111111</td>
</tr>
<tr>
<td>0.175–0.180</td>
<td>$LS^5$</td>
<td>7</td>
<td>8</td>
<td>1/8</td>
<td>0.125000</td>
</tr>
<tr>
<td>0.200</td>
<td>$LS^4$</td>
<td>6</td>
<td>7</td>
<td>1/7</td>
<td>0.142857</td>
</tr>
<tr>
<td>0.220</td>
<td>$LS^3.4LS^4$</td>
<td>11</td>
<td>13</td>
<td>2/13</td>
<td>0.158346</td>
</tr>
<tr>
<td>0.223</td>
<td>$(LS^3)^2LS^4$</td>
<td>16</td>
<td>19</td>
<td>3/19</td>
<td>0.157855</td>
</tr>
<tr>
<td>0.225</td>
<td>$(LS^3)^2LS^3$</td>
<td>21</td>
<td>25</td>
<td>4/25</td>
<td>0.160000</td>
</tr>
<tr>
<td>0.226</td>
<td>$(LS^3)^4LS^4$</td>
<td>26</td>
<td>31</td>
<td>5/31</td>
<td>0.161290</td>
</tr>
<tr>
<td>0.230–0.255</td>
<td>$LS^3$</td>
<td>5</td>
<td>6</td>
<td>1/6</td>
<td>0.166667</td>
</tr>
<tr>
<td>0.238–0.338</td>
<td>$LS^2...LS^3$</td>
<td>4...5</td>
<td>5</td>
<td>1/5</td>
<td>0.200000</td>
</tr>
<tr>
<td>0.392–0.550</td>
<td>$LS...LS^2$</td>
<td>3...4</td>
<td>4</td>
<td>1/4</td>
<td>0.250000</td>
</tr>
<tr>
<td>0.600</td>
<td>$LSLS^2$</td>
<td>7</td>
<td>7</td>
<td>2/7</td>
<td>0.285714</td>
</tr>
<tr>
<td>0.630</td>
<td>$(LS)^2LS^2$</td>
<td>10</td>
<td>10</td>
<td>3/10</td>
<td>0.300000</td>
</tr>
<tr>
<td>0.650</td>
<td>$(LS)^3LS^2$</td>
<td>13</td>
<td>13</td>
<td>4/13</td>
<td>0.307692</td>
</tr>
<tr>
<td>0.655</td>
<td>$(LS)^4LS^2$</td>
<td>16</td>
<td>16</td>
<td>5/16</td>
<td>0.312500</td>
</tr>
<tr>
<td>0.660</td>
<td>$(LS)^5LS^2$</td>
<td>19</td>
<td>19</td>
<td>6/19</td>
<td>0.315790</td>
</tr>
<tr>
<td>0.663</td>
<td>$(LS)^6LS^2$</td>
<td>22</td>
<td>22</td>
<td>7/22</td>
<td>0.318182</td>
</tr>
<tr>
<td>0.664</td>
<td>$(LS)^7LS^2$</td>
<td>25</td>
<td>25</td>
<td>8/25</td>
<td>0.320000</td>
</tr>
<tr>
<td>0.665</td>
<td>$(LS)^7LS^2)^2LS^3LS^2$</td>
<td>78</td>
<td>78</td>
<td>25/78</td>
<td>0.320513</td>
</tr>
<tr>
<td>0.666</td>
<td>$(LS)^8LS^2$</td>
<td>28</td>
<td>28</td>
<td>9/28</td>
<td>0.321429</td>
</tr>
<tr>
<td>0.667</td>
<td>$(LS)^9LS^2$</td>
<td>31</td>
<td>31</td>
<td>10/31</td>
<td>0.322581</td>
</tr>
<tr>
<td>0.669</td>
<td>$(LS)^{14}S$</td>
<td>43</td>
<td>43</td>
<td>14/43</td>
<td>0.325581</td>
</tr>
<tr>
<td>0.671</td>
<td>$(LS)^{27}S$</td>
<td>82</td>
<td>82</td>
<td>27/82</td>
<td>0.329268</td>
</tr>
<tr>
<td>0.672–1.0</td>
<td>$LS$</td>
<td>3</td>
<td>3</td>
<td>1/3</td>
<td>0.333333</td>
</tr>
<tr>
<td>2.0</td>
<td>$L$</td>
<td>2</td>
<td>2</td>
<td>1/2</td>
<td>0.500000</td>
</tr>
</tbody>
</table>

However, this pattern disappears at $\varepsilon = \varepsilon_1$ — for our main example $\mu = 0.0014$, $\varepsilon_1 \approx 0.089$ — and for slightly higher values of $\varepsilon$ the patterns are of the form $LS^n$ with large values of $n$. This can be explained by a saddle-node bifurcation at $\varepsilon_1$ where the fixed points meet and vanish again, since the behavior is typical for the intermittency associated with such a bifurcation: Most of the time is spent drifting slowly across an interval $C \subset S^1$ where the critical points previously existed.

In $C$ the dynamics is dominated by the forcing, since the autonomous part is small. When the orbit leaves $C$ the autonomous dynamics dominate, and the effect of the large oscillation is to move the state around the circle and into $C$ again from the other side. See Fig. 3.
large values of $\varepsilon$, the patterns without shoulders dominate, and Eq. (7) holds.

For low values of $\varepsilon$ Eq. (7) also holds if the number of shoulders is included in the $S$'s. The patterns $LS^n$ in the range $0.05 < \varepsilon < 0.200$ have one shoulder and hence a modified pattern $LS^{n+1}$. The patterns $(LS^3)^nLS^4$ in the range $0.200 < \varepsilon < 0.226$ have, counting shoulders, the pattern $(LS^4)^nLS^5$. See Fig. 4(c).

For $\mu < 0$, almost the same behavior as described above is observed. The only difference is that for small $\varepsilon$, only quasi-periodic solutions are found. However, at some $\varepsilon = \varepsilon_2$ a stable periodic solution with pattern $S$ is created in a saddle-node bifurcation, and the complete scenario of patterns of periodic solutions described above for $\mu > 0$ emerges.

### 3.2. Rotation numbers

Recall that if the circle is parametrized by an angular parameter $\theta \in [0, 1]$, the rotation number $\rho$ of the map $g$ is defined by

$$\rho = \lim_{n \to \infty} \frac{G^n(\theta) - \theta}{n},$$

where $G$ is a lift of $g$, i.e. with points identified modulo 1 [Guckenheimer & Holmes, 1983]. The rotation number exists for all $\theta$ and is independent of $\theta$. The rotation number is the average rotation during an iteration, and a key result on circle maps, on which our analysis hinges, is that the rotation number is rational if and only if the map has a periodic orbit.

Consider now a parametrized family of circle diffeomorphisms $\{g_\alpha | \alpha \in A\}$, where $A$ is some interval. If the function $\alpha \mapsto \rho(\alpha)$ has the following properties:

1. $\rho(\alpha)$ is continuous
2. $\rho(\alpha)$ is nondecreasing
3. $\rho(\alpha)$ is locally constant at each rational $\alpha$ and nonconstant at each irrational $\alpha$
4. The set $\{\alpha \in A | \rho(\alpha) \text{ is irrational}\}$ is nowhere dense in $A$,

the graph of $\rho$ is a Devil’s staircase.

Loosely speaking, the existence of a Devil’s staircase implies that for almost all parameter values a periodic solution exists, and it is very difficult for a computer to find a parameter value that gives a non-periodic solution. Since the minimal period of a periodic solution equals the denominator in the rotation number, and irreducible rationals with arbitrarily large denominators can be found in the interval $\rho(A)$, periodic solutions with arbitrarily
long periods exist. These are exactly the properties stated as basic observation 1.

We propose that the rotation number for the circle map for the Oregonator is a Devil’s staircase when \( \varepsilon \) or \( T_{in} \) are considered the parameter.

Even if we do not have an analytic expression for the circle map \( g \) we claim that the rotation number can be calculated directly from simulations. The orbits we consider have \( P \geq 1 \) and during the sampling time \( T_{in} \) of the stroboscopic map a point of the limit cycle has moved less than one complete turn of the circle. However, each time a large spike in the continuous time trace is produced, a complete turn of the circle has been accomplished. Since the rotation number is the average rotation angle during a period, we get

\[
\rho = \frac{\text{number of } L's \text{ in a period}}{\text{reduced period } P}. \tag{9}
\]

The numerator is the number of turns of the circle computed during a period and the denominator is the number of iterations of \( g \) used to complete the periodic motion.

Table 2 shows that the rotation number computed this way is indeed a nondecreasing function of \( \varepsilon \), hence substantiating our claim.

It should be noted that the existence of a Devil’s staircase in a family of circle diffeomorphisms is a common feature. For instance, it occurs for every family of the form \( f_\alpha = f + \alpha \), where \( f \) is a fixed homeomorphism of the circle for which no iterate is the identity [de Melo & van Strien, 1993]. These include the standard map originally studied by Arnol’d,

\[
f_\alpha(\theta) = \theta + a \sin(2\pi \theta) + \alpha, \quad 0 < a < \frac{1}{2\pi}. \tag{10}
\]

Furthermore, from the continuity of \( \rho \) the period adding property stated as the second basic observation is easily explained:

**Proposition.** Assume that the parameter values \( \alpha_1 < \alpha_2 \) give rise to periodic orbits with periods \( q_1 \) and \( q_2 \) respectively, \( q_1 \neq q_2 \). There exists an \( \alpha_3 \in (\alpha_1, \alpha_2) \) for which \( g_\alpha \) has a periodic orbit with period \( q_3 = q_1 + q_2 \).
Proof. Recall that the denominator in a rational irreducible rotation number determines the period of the stable, periodic solution. Hence there exists integers $p_1$ and $p_2$ such that
\[ \rho(\alpha_1) = \frac{p_1}{q_1}, \quad \rho(\alpha_2) = \frac{p_2}{q_2}, \]
both irreducible. Since $\rho(\alpha)$ is nondecreasing, we know that
\[ \frac{p_1}{q_1} \leq \frac{p_2}{q_2}, \]
where equality is ruled out by irreducibility. Now, the Farey sum of the two fractions, i.e.
\[ \frac{p_1}{q_1} \oplus \frac{p_2}{q_2} = \frac{p_1 + p_2}{q_1 + q_2} \]
fulfills
\[ \frac{p_1}{q_1} < \frac{p_1}{q_1} \oplus \frac{p_2}{q_2} < \frac{p_2}{q_2}, \]
and it follows from the continuity of $\rho$ that there exists an $\alpha_3 \in (\alpha_1, \alpha_2)$ for which
\[ \rho(\alpha_3) = \frac{p_1 + p_2}{q_1 + q_2}. \]
When the rotation number is rational with denominator $q_1 + q_2$, there is a periodic solution with period $q_1 + q_2$. ■

3.3. Scaling

The fractal structure of a Devil’s staircase indicates that scaling behavior is to be expected. Indeed, numerical simulations of the standard map (10) indicate that the range of the parameter interval in which $q$-periodic solutions exist scale as $q^{-3}$ [Cvitanović et al., 1985; Ecke et al., 1989]. This is confirmed by analytical results which we briefly describe now.

Two irreducible rationals $p_1/q_1$ and $p_2/q_2$ are Farey neighbors if
\[ |p_1q_2 - p_2q_1| = 1. \]  
(11)

Take two arbitrary Farey neighbors $p_1/q_1$ and $p_2/q_2$ and construct the sequence
\[ u_n = \frac{np_1 + p_2}{nq_1 + q_2}, \quad n \in \mathbb{N}, \]
and denote by $R_n$ the length of the parameter interval in which the circle map has a stable periodic orbit with rotation number $u_n$. Jonker [1990] proves under very general assumptions that there exist positive constants $C_1$ and $C_2$ independent of $n$ such that
\[ C_1(nq_1 + q_2)^{-3} < R_n < C_2(nq_1 + q_2)^{-3} \]  
(13)
for all $n$. The constants $C_1$ and $C_2$ depend of course on the choice of Farey neighbors, but Graczyk [1991] shows that $C_1$ and $C_2$ can be chosen such that the inequalities (13) hold for arbitrary Farey neighbors $p_1/q_1$ and $p_2/q_2$.

These results imply that the exponent in Eq. (3) should be $a = 1/3$. The numerically obtained exponents between 0.26 and 0.44 are in good agreement with the theoretical result. It should be noted that some uncertainty is due to the fact that simulations on the chemical oscillator only have been possible for relatively small $n$.

To see how this scaling works, consider the sequence of periodic responses with the pattern $(LS)^nL^2S$ which is one of the cases studied by Markman and Bar-Eli [1994a]. This corresponds to a response of length $3n + 5$ with $n + 2$ large oscillations per period. According to (9), the rotation number of such a response is
\[ \rho = \frac{n + 2}{3n + 5} = \frac{n \cdot 1 + 2}{n \cdot 3 + 5} \]
This is the $u_n$-sequence associated to the rationals 1/3 and 2/5, which are Farey neighbors according to (11). Hence there exists constants $C_1$ and $C_2$ according to (13), such that
\[ C_1(3n + 5)^{-3} < \text{range}((LS)^nL^2S) < C_2(3n + 5)^{-3}. \]

Finally, the scaling (4) is explained by noting that a periodic solution disappear at the end of a step in a saddle-node bifurcation of some iterate of $g$. It is well known that this is associated with intermittency, as described in Sec. 3.1. Simple arguments [Guckenheimer & Holmes, 1983] show that the time spent by an orbit in intermittency scales like Eq. (4) with $c = 1/2$. The numerically obtained values of $c$ between 0.51 and 0.63 are again in good agreement with the theoretical prediction.

4. Conclusions

We have demonstrated that a number of interesting phenomena in the Oregonator can be attributed to the presence of a Devil’s staircase in a circle map that governs the dynamics of the system. The conclusions, however, will be valid for a much larger class of periodically forced systems. The analysis is essentially based on the presence of a saddle-loop
bifurcation in the unforced system and it must expected that similar behavior occurs in other situations where this bifurcation is present. However, a technical condition on the attractivity of the invariant set $\Gamma$ is crucial for the analysis. If this condition is not met, much more complicated dynamics, including chaos, may occur.

The Farey arithmetic has previously been used to describe dynamic features of the Belousov–Zhabotinskii reaction. For example Maselko and Swinney [1986, 1987] have both numerically and experimentally found similar concatenation rules for patterns of small and large oscillations. However, in these studies no external forcing was applied and the patterns where associated with dynamics on invariant tori in the phase space, hence being of a quite different nature than the case considered here.

References


