Numerical computation of Lyapunov exponents in discontinuous maps implicitly defined

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Abstract

This paper describes some numerical techniques to compute the Lyapunov exponents of discontinuous maps implicitly defined. Special attention is paid to the case of maps generated by stick-slip systems but the techniques presented in the paper are applicable to wider classes of maps. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

A multi-dimensional map is usually defined by a difference equation [1,2]:

\[ x_{n+1} = f(x_n), \]

where \( x \) is an \( n \)-dimensional vector and \( f \) is a non-linear transformation. The Poincaré map of any continuous dynamical system whose phase space dimension is greater than two gives one of the most common examples of a multi-dimensional map. In general the equation of the Poincaré map is not explicitly known but the successive points of the map are computed by means of the integration of the original continuous system. If the function \( f \) in (1) is smooth, or if the underlying continuous dynamical system is smooth, then the multi-dimensional map is smooth and many theoretical results are available about existence and stability of fixed points and limit cycles or about the existence of Lyapunov exponents, etc., etc. [1–3].

Many of these theoretical results are based on the examination of the Jacobian matrix, which in the case of a two-dimensional map:

\[ x_{n+1} = f(x_n, y_n), \]
\[ y_{n+1} = g(x_n, y_n) \]

is given by the expression:

\[ Df = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}. \]

In the field of stick-slip systems the introduction of low-dimensional implicitly defined maps has often proven very useful in the understanding of the system dynamics [4–11]. The successive iterates of the maps are computed by means of the integration, in continuous time, of a discontinuous, structurally variable system [5]. The equations of these maps are not known and in general no expression for the Jacobian matrix is therefore available. In spite of the fact that the maps described in this paper have been introduced

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for stick-slip systems, there is no apparent reason to restrict the use of the numerical techniques described below to these dynamic systems and therefore we believe that the content of this paper can be applied to wider classes of dynamic systems.

2. Maps generated by stick-slip systems

We briefly recall how stick-slip systems may generate one- or multi-dimensional maps [9]. A two-block system is considered (see Fig. 1(a)): in this system each single block may stick on the belt or slip with respect to the belt, which moves with constant velocity $V_{dr}$. If the driving velocity is small, in all possible steady state motions time intervals will exist in which the two blocks are simultaneously sticking on the belt: such a phase of motion is called global stick phase. Moreover in the case of small driving velocity all motions in which one or both blocks are slipping will be attracted by a steady state motion with global stick phases, after that a possible initial transient has died out. During a global stick phase the distance between the two blocks is fixed therefore the dynamics of the system may be characterized by the constant value of a scalar variable $d$:

$$d = x_2 - x_1,$$  \hspace{1cm} (4)

where $x_i$ is the displacement of the $i$th block from a reference configuration in which all springs assume their natural length and $d$ is the relative displacement between the two blocks. In the plane of the displacement variables it is possible to locate a region where the global stick phases have to lie:

$$-F_{s1} < k_1 x_1 + k_{12} (x_1 - x_2) < F_{s1}, \hspace{1cm} (5)$$

$$-F_{s2} < k_2 x_2 + k_{12} (x_2 - x_1) < F_{s2},$$

where $F_{si}$ is the maximum static friction force acting on the $i$th block, $k_i$ the stiffness of the spring connecting the $i$th block to a fixed support and $k_{12}$ the stiffness of the coupling spring. The boundary of this quadrilateral region (Fig. 1(b)) contains the points $A$ and $C$ corresponding to the two limit values $d_{\text{min}}, d_{\text{max}}$ for the variable $d$. Any motion of the system generates an infinite sequence of values of the variable $d = d_1, d_2, \ldots, d_n, \ldots$, which can be interpreted as a one-dimensional map expressing $d_{k+1}$ as a function of $d_k$:

$$d_{k+1} = f(d_k).$$  \hspace{1cm} (6)

The map is defined in the interval $d_{\text{min}} < d < d_{\text{max}}$. In a periodic motion an integer $j$ exists such that $d_{j+1} = d_1$, in this case the map too is periodic, of period $j$, whereas if a motion is not periodic it will generate a non-periodic map.

A typical shape of this map is shown in Fig. 2, where the thick lines correspond to non-periodic attractors and the thin lines to transient trajectories. This figure illustrates some important features of the map: discontinuous definition of the motion equations and non-smoothness of the forces acting on the blocks generate a piece-wise continuous map with well-localized discontinuities.

In the case of three blocks (Fig. 3(a)) and small driving velocity, any global stick phase is characterized by two independent constant values:

$$d_1 = x_2 - x_1,$$  \hspace{1cm} (7)

$$d_2 = x_3 - x_1.$$
Fig. 2. One-dimensional map generated by the system of Fig. 1 with $V_{dr} = 0.0815$. The map has two non-periodic attractors indicated by the thick lines. The intervals along the horizontal axis indicated by the letter $a$ are the basin of attraction of the $A_1$ attractor. The remaining parts of the interval $d_{\text{min}} < d < d_{\text{max}}$ are the basin of attraction of the $A_2$ attractor. Parameter values: $m_1 = m_2 = 1.0, k_1 = k_2 = 1.0, k_{12} = 1.2$, $F_{s1} = 1.0, F_{s2} = 1.3, \Rightarrow d_{\text{max}} = d_{\text{min}} = 0.6764$, friction characteristic as in [9].

$$-F_{s1} \leq k_1 x_1 + k_{12} (x_1 - x_2) + k_{13} (x_1 - x_3) \leq F_{s1},$$
$$-F_{s2} \leq k_2 x_2 + k_{12} (x_2 - x_1) + k_{23} (x_2 - x_3) \leq F_{s2},$$
$$-F_{s3} \leq k_3 x_3 + k_{13} (x_3 - x_1) + k_{23} (x_3 - x_2) \leq F_{s3},$$

(8)

where the meaning of the symbols is the same as in Eq. (5). The projection of this prism (Fig. 3(b)) of the three-dimensional displacement space $(x_1, x_2, x_3)$ on the space of the two variables $d_1, d_2$, gives rise to an irregular hexagon as the one shown in Fig. 3(c) which is the field where the following two-dimensional map is defined:

$$d_{1,k+1} = f_1(d_{1k}, d_{2k}),$$
$$d_{2,k+1} = f_2(d_{1k}, d_{2k}).$$

(9)

Assuming that the map possesses the same features as the one-dimensional case, it can be used to find the attractors of the dynamic system and their basins [11]. In fact in this case it is not possible to visualize the shape of the map but it is possible to visualize the discontinuity lines in the definition field of the map. In Fig. 4 we show the discontinuity lines for four cases of the two-dimensional map. They indicate the fact that if we choose two points $F$ and $P$ (Fig. 5), in the definition field of the map, which are not separated by a discontinuity line and such that $d_{1F} \equiv d_{1P}$, $d_{2F} \equiv d_{2P}$ then their images are such that $(f_1(d_{1F}, d_{2F}), f_2(d_{1F}, d_{2F})) \equiv (f_1(d_{1P}, d_{2P}), f_2(d_{1P}, d_{2P}))$. This is not true if we...
choose the points \( F \) and \( Q \), that are separated by a discontinuity line, in this case the proximity of the points is not sufficient to guarantee the proximity of their images under the action of the map.

The discontinuity lines are found precisely in this way: a grid of points is chosen in the definition field of the map and for each of them the first iterate of the map is computed, then for each point \( F \) of the grid the following two ratios are evaluated:

\[
\begin{align*}
    r_{PF} &= \frac{\| (f_1 F, f_2 F) - (f_1 P, f_2 P) \|}{\| (d_1 P, d_2 P) - (d_1 F, d_2 F) \|}, \\
    r_{QF} &= \frac{\| (f_1 Q, f_2 Q) - (f_1 F, f_2 F) \|}{\| (d_1 Q, d_2 Q) - (d_1 F, d_2 F) \|}.
\end{align*}
\]

(10)
Where $P$ is the point of the grid that follows $F$ in the vertical direction, $Q$ is the point of the grid that follows $F$ in the horizontal direction, $f_1F = f_1(d_1F, d_2F)$, $f_2F = f_2(d_1F, d_2F)$, etc. If the value of one or both of these two ratios is 'large', then the point $F$ is assumed to belong to a discontinuity line.

If a chain of $m$ blocks is taken into consideration the sequence of global stick phases generates the $(m - 1)$-dimensional map of the relative displacements between pair of blocks.

3. Lyapunov exponents

Given a discrete (or continuous) dynamical system in an $m$-dimensional phase space we follow the evolution of an infinitesimal $m$-sphere of post-transient initial conditions under the action of the dynamical system. The infinitesimal $m$-sphere becomes an infinitesimal $m$-ellipsoid. A generic one-dimensional Lyapunov exponent is defined as [12,13]:

$$
\lambda_i^p = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \ln \frac{p_{i,j}}{p}, \quad i = 1, 2, \ldots, m,
$$

(11)

where $p_{i,j}$ is the length of the $i$th ellipsoidal axis after $j$ iterations and $p$ the initial diameter of the $m$-sphere. It is possible to show that for a randomly chosen initial condition on the $m$-sphere one obtains the largest exponent of order one.

Similarly it is possible to define the Lyapunov exponents of order $p$ ($p < m$) as:

$$
\Lambda^p = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \ln \frac{Vol_{i,j}}{Vol_0},
$$

$$
i = 1, 2, \ldots, m! \frac{1}{p!(m-p)!},
$$

(12)

where $!$ = 1, $Vol_0$ is the initial volume of a $p$-sphere and $Vol_{i,j}$ is the volume of the corresponding $p$-ellipsoid generated by $j$ iterations of the dynamic system. In a $m$-dimensional system there exists one Lyapunov exponent of order $m$. It is also possible to show that each Lyapunov exponent of order $p$ is the sum of $p$ exponents of order one and that, for a randomly chosen initial $p$-sphere of volume $Vol_0$, one obtains the sum of the $p$ largest exponents [2]:

$$
\Lambda^p = \lambda_1^p + \lambda_2^p + \cdots + \lambda_p^p.
$$

(13)

In the case of smooth maps the evaluation of the Lyapunov exponents generally requires the application of numerical techniques based on the simultaneous integration of the dynamical system and of its linearized equations. In the most commonly used numerical techniques it is necessary to introduce a ‘fiducial’ trajectory [13] which is defined by the dynamics of the system on some post-transient initial condition, this trajectory is indicated in the following by $F_n$ (where $n = 1, 2, 3 \ldots$ indicates the number of iterations of the map). Moreover the orbital structure of an $m$-dimensional map can be investigated by introducing $m$ perturbed trajectories, whose initial conditions define an arbitrarily oriented frame of $m$-orthogonal vectors.

In the case of dynamic systems with discontinuities the linearized equations have to be supplemented with transition conditions at the instants of discontinuities [14]. This method can be applied to usual discontinuous systems as stick-slip or impacting systems when the location of the discontinuities is known as it is usually the case if the equations of motion in continuous time are integrated.

The stick-slip maps are numerically computed because their form is not explicitly known therefore the explicit form of their Jacobian matrix is unknown and their Lyapunov exponents may be numerically computed by means of a finite difference technique. Let us focus our attention on the 2D map defined by Eqs. (9). The Lyapunov exponents quantify the expanding or contracting nature of the flow in the phase space [12, 13]. Under the action of the map distances in the phase space along the fastest growing direction vary as $e^{\lambda_1 n}$. Similarly the area of a mapped portion of the plane varies as $e^{(\lambda_1 + \lambda_2) n}$. In Fig. 6(a) the point $F_n$ indicates the $n$th iterate of a fiducial trajectory, $G_{1,n}$ a perturbed trajectory, at a distance $\rho$ from $F_n$, free to seek out the fastest growing direction, and $G_{2,n}$ a perturbed trajectory chosen at the same distance $\rho$ in the direction normal to the line connecting $F_n$ to $G_{1,n}$. The three points are iterated so that $F_{n+1}, G_{1,n+1}, G_{2,n+1}$ are found (see Fig. 6(b)). An approximation to the Lyapunov exponents can then be numerically computed as:

$$
\lambda_{1,\text{num}} = \frac{1}{n+1} \sum_{i=1}^{n+1} \ln \frac{\rho_i}{\rho},
$$

(14.a)
Fig. 6. Calculation of the Lyapunov exponents neglecting the possibility of having discontinuity lines between fiducial trajectory and perturbed trajectories.

\[ A_{\text{num}}^2 = \lambda_{1,\text{num}}^1 + \lambda_{2,\text{num}}^1 = \frac{1}{n+1} \sum_{i=1}^{n+1} \ln \frac{A_i}{\rho^2}, \]  
(14.b)

where \( \rho_{n+1} \) is the distance between \( F_{n+1} \) and \( G_{1,n+1} \) and \( A_{n+1} \) is the area shown in Fig. 6(b). This process has to be repeated for a large number of iterations of the fiducial trajectory, therefore the point \( H_{1,n+1} \) is chosen on the line connecting \( F_{n+1} \) to \( G_{1,n+1} \) at a distance \( \rho \) from \( F_{n+1} \) and the Gram–Schmidt orthogonalization is used to locate the point \( H_{2,n+1} \) at the same distance. The three points \( F_{n+1}, H_{1,n+1}, H_{2,n+1} \) are iterated again and again so that the correct values of the Lyapunov exponents, which are defined as limit for \( n \) that tends to \( \infty \) of Eqs. (14), can be accurately approximated.

The value of the distance \( \rho \) is of crucial importance for the correct evaluation of the Lyapunov exponents. In fact \( \rho \) has to be small enough to capture the local behaviour of the map but not too small to avoid numerical instabilities.

The numerical procedure outlined above assumes that fiducial trajectory and perturbed trajectories lie on the same branch of the discontinuous map but it gives meaningless values if that hypothesis is not respected. This can be easily understood in the graphical representation of a one-dimensional map, as shown in Fig. 7, where \( F \) is the fiducial trajectory and \( G \) the perturbed trajectory. In this case the ratio \( |F_{n+1} - G_{n+1}|/|F_n - G_n| \) would not represent the stretch of an infinitesimal neighbourhood around \( F_n \) and would give a meaningless value. It is clear that in the case of large \( \rho \) the probability of having fiducial trajectory and perturbed trajectory on different branches of the map is high whereas for small \( \rho \) such a probability is low, therefore it is reasonable to assume that the influence of the discontinuity of the map on the computation of

Fig. 7. Problem in the computation of Lyapunov exponents if fiducial and perturbed trajectories lie on different branches of a discontinuous map. This figure is an enlarged view of a portion of Fig. 2.
Fig. 8. How to avoid the use of squared powers in the computation of second order Lyapunov exponents.

the Lyapunov exponent decreases with $\rho$. The same problem affects the multidimensional maps if the fiducial trajectory and one or more of the perturbed trajectories lie on separate branches of the map, i.e. if they are separated by a discontinuity line as points $F$ and $Q$ in Fig. 5. This problem may be (at least partially) solved by introducing a check as the one described by Eqs. (10): if the images of one or more perturbed trajectories are ‘too far’ from the image of the fiducial trajectory then the procedure assumes that those perturbed trajectories do not lie on the same branch of the map as the fiducial trajectory. In this case the values of the images of the perturbed trajectories are discarded and recomputed adopting a smaller $\rho$ and the process is repeated until all the images of the perturbed trajectories are ‘close’ to the image of the fiducial trajectory. This method can be applied in the case of ‘purely’ one-dimensional discontinuity lines but it could fail in the case of ‘complex’ discontinuity lines, as clarified in the next remark.

In the case of multi-dimensional map the computation of the order $p$ Lyapunov exponent presents some additional numerical difficulties with respect to the computation of the largest order one exponent. In fact, if the distance $\rho$ has to be small so that fiducial and perturbed trajectories lie on the same branch of the map, there may be some numerical problems in computing the ratio $(A_i/\rho^2)$ of relation (14.b): if $\rho$ is of the order $10^{-8}-10^{-10}$ the ratio $(A_i/\rho^2)$ is an operation between numbers in the range $10^{-16}-10^{-20}$ that may be very inaccurate. The accuracy of this com-

Fig. 9. Region where the computation of the Lyapunov exponents presents unresolved difficulties. This figure is an enlarged view of a portion of Fig. 4(d).
Fig. 10. Lyapunov exponents of a 2D map generated by the 3 block system. Parameters: \( m_1 = m_2 = m_3 = 1.0, k_1 = k_2 = k_3 = 1.0, k_{12} = k_{13} = k_{23} = 1.0, F_{s1} = 1.0, F_{s2} = 1.14, F_{s3} = 1.18 \), friction characteristic as in [9].

The computation can be highly improved by computing the aforementioned ratio as product between two ratios as:

\[
\frac{A_i}{\rho^2} = \left( \frac{\|G_{1,i} - F_i\|}{\rho} \right) \left( \frac{l}{\rho} \right),
\]

where the length \( l \) is defined in Fig. 8. In this way two ratios between numbers in the range \( 10^{-8} - 10^{-10} \) are computed, each ratio will usually be in the range \( 10^{-3} \) and the product of these two ratios will generally give an accurate result.

Similar operations are more and more necessary for Lyapunov exponents of order higher than two.

**Remark.** Fig. 9 shows a region in the definition field of the two-dimensional map (9) where the choice of a small \( \rho \) would not improve the accuracy of the computation because of the complexity of the discontinuity lines. However in this specific case the computation of the Lyapunov exponents is not affected by the irregularity of the discontinuity lines because there is no intersection between the attractors and these regions. Our procedure would fail in the hypothetical case of a chaotic attractor covering a region of irregular discontinuity lines.

### 4. Results

The numerical techniques described in this paper have been applied to the computation of the Lyapunov exponents of a two-dimensional map generated by a three-block stick-slip system [11]. According to the results of our computations, the dynamics of the map, and therefore of the underlying mechanical system, can be classified as (Fig. 10):

- Hyper-chaotic, where \( \lambda^1_1 > \lambda^1_2 > 0 \).
- Kaplan–Yorke chaotic [15], where \( \lambda^1_1 > 0 > \lambda^1_2 \) and \( \Delta^2 = \lambda^1_1 + \lambda^1_2 > 0 \).
- Chaotic, where \( \lambda^1_1 > 0 > \lambda^1_2 \) and \( \Delta^2 = \lambda^1_1 + \lambda^1_2 < 0 \).
- Periodic, where \( 0 > \lambda^1_1 > \lambda^1_2 \).

Fig. 10 shows the evolution of the Lyapunov exponents as \( V_{dr} \) is varied and Fig. 11 shows some qualitatively different map dynamics corresponding to the previous definitions. It is apparent that the qualitative differences among the maps confirm the Lyapunov exponent computations. The computation of these two Lyapunov exponents fully classifies the dynamics of the three block system because the values of the four other exponents are already known: \( \lambda_6 = -\infty, \lambda_5 = -\infty, \lambda_4 = -\infty, \lambda_3 = 0 \) [11,14]. Moreover it is clearly faster than the computation of the exponents carried out with the method presented in [14], but it is also evident that the method presented in [14] is of more general applicability.
Fig. 11. Dynamics of some cases of the 2D map of Fig. 10 for different values of the driving velocity. (a) Hyper-chaotic motion, where $\lambda_1^2 > \lambda_2^1 > 0$; (b) Kaplan–Yorke chaotic motion, where $\lambda_1^1 > 0 > \lambda_1^2$ and $A^2 = \lambda_1^1 + \lambda_1^2 > 0$, chaotic motion, where $\lambda_1^1 > 0 > \lambda_1^2$ and $A^2 = \lambda_1^1 + \lambda_1^2 < 0$. The enlarged views show that a Kaplan–Yorke chaotic motion is similar to a ‘noisy’ chaotic motion. The three figures (a), (b), (c) contain the same number of post transient points, 50,000.

Conclusions

Some low dimensional maps generated by mechanical stick-slip systems have been introduced as examples to show some difficulties arising in the computation of Lyapunov exponents of discontinuous maps implicitly defined. The paper presents some numerical techniques to overcome such difficulties.

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