Fresnel Type Path Integral for a Kind of Stochastic Schrödinger Equations Driven by Semimartingales

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Abstract

We prove a path integral representation in configuration space for the solution of a type of stochastic Schrödinger equation driven by continuous semimartingales. We follow the approach of path integration established by Itô, Albeverio and Høegh-Krohn. The equation we consider is formed by the classical Schrödinger equation plus the stochastic term $K(t, x) \Psi_t(x) \circ df_t$, where $f_t$ is a continuous semimartingale. As an application of our result we solve the equation for a stochastic quantum harmonic oscillator. To solve this equation we use two methods, one is by discretising the path integral and the second method is via a change of variables.

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1 Introduction

In 1948, R. Feynman formally expressed the solution of the Schrödinger equation in terms of an ill-defined integral over a space of paths. Since then, an enormous amount of work has been devoted to finding a rigorous definition of such path integral, and several definitions have been proposed (see references in [1]). One particularly successful approach to Feynman path integrals has been that suggested by Itô, Albeverio and Høegh-Krohn in [1]. Following the work of Albeverio et al., in 1996 T. Zastawniak [5] proved an extension to the stochastic case of this path integral representation in configuration space. He considered a kind of stochastic Schrödinger equation driven by Brownian motion and proved that a path integral representation existed for the solution of this equation. Further extensions of path integrals on phase spaces have been recently done by Truman and Zastawniak [3]. In this paper we prove an analogous result of that presented by Zastawniak in [5] for the general case when the drift of the stochastic equation is a continuous semimartingale. Only minor changes are needed in the proof of Zastawniak to achieve our extension, but full details are spelt out in our exposition.

We start by defining the type of stochastic equation to study and giving the setup of the Albeverio et al. approach to Feynman path integral. Then we state and proof our main result, and finally apply it to the case of a stochastic harmonic oscillator.

2 Fresnel Type Path Integration

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $\{\mathcal{F}_t\}_{t \geq 0}$ a standard filtration and $f : [0, \infty) \times \Omega \to \mathbb{R}$ a continuous semimartingale. We consider the stochastic Schrödinger equation

$$i \, d\Psi_t(x) = \left( -\frac{1}{2} \Delta_x + V(t, x) \right) \Psi_t(x) \, dt + K(t, x) \Psi_t(x) \circ df_t, \quad (1)$$

where $t > 0$, $x \in \mathbb{R}^d$ and $V(t, x)$ and $K(t, x)$ are real-valued functions. The symbol $\circ$ indicates that the stochastic differential is taken in the Stratonovich sense. The term $K(t, x) \Psi_t(x) \circ df_t$ can be regarded as an added multiplicative noise on the wave function $\Psi_t(x)$. For notational simplicity we have chosen units so that $\hbar = 1$ and our particle has mass $m = 1$. 

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Let us assume there exists a solution to equation (1) with initial condition
\[ \Psi_t(x)|_{t=0} = \Psi_0(x). \]

Under some further assumptions we shall express this solution as an integral over a space of paths
\[ \Psi_t(x) = \int_{\text{paths}} e^{iS(x+\gamma)} \Psi_0(x + \gamma_0) \, d\gamma \quad (2) \]
where \( \gamma : [0,t] \rightarrow \mathbb{R}^d \) is a path and the action functional \( S(x + \gamma) \) is given by
\[ S(x + \gamma) = \frac{1}{2} \int_0^t |\dot{\gamma}(s)|^2 \, ds - \int_0^t V(s, x + \gamma(s)) \, ds - \int_0^t K(s, x + \gamma(s)) \, ds \]
and \( \gamma_0 := \gamma(0) \). Observe last integral is an Itô integral.

We will now be more precise about above path integral representation (2). The main ideas come from K. Itô [2], S. Albeverio and R. Høegh-Krohn [1] and T. Zastawniak [5].

2.1 The Space of Paths

We first introduce the Hilbert space of paths where the functional integration is carried over. For any fixed \( t > 0 \), let \( \mathcal{H}_t \) be the Hilbert space of continuous paths \( \gamma : [0,t] \rightarrow \mathbb{R}^d \) with \( \gamma(t) = 0 \) and such that each component of \( \dot{\gamma} \) is square integrable. A path is interpreted as a generalised function and its derivative is then understood in the weak sense. The inner product in \( \mathcal{H}_t \) is defined by
\[ (\gamma, \eta)_t := \int_0^t (\dot{\gamma}(s), \dot{\eta}(s)) \, ds, \]
where \( (x, y) = x \cdot y \) is the usual Euclidean inner product of \( x, y \) in \( \mathbb{R}^d \). This Hilbert space has the reproducing kernel \( G : [0, t] \times [0, t] \rightarrow \mathbb{R}^d \), where the \( k \)-th component of \( G \) is given by \( G^k(\sigma, \tau) = t - \sigma \vee \tau \). That is, for fixed \( \sigma \), the path \( \tau \mapsto G(\sigma, \tau) \) belongs to \( \mathcal{H}_t \) and is such that for any \( \gamma \in \mathcal{H}_t \),
\[ \gamma^k(\sigma) = (\gamma^k(\cdot), G^k(\sigma, \cdot))_t, \]
where the upper index denotes component. The existence of such a reproducing kernel will be a key feature in our ensuing calculations.
2.2 The Space of measures

Now let $\mathcal{M}(\mathcal{H}_t)$ be the Banach algebra of bounded complex-valued measures on $\mathcal{H}_t$. The norm on this space is given by the total variation norm, that is, for $\mu \in \mathcal{M}(\mathcal{H}_t)$, we define $\|\mu\| := |\mu|(\mathcal{H}_t)$. The operations in $\mathcal{M}(\mathcal{H}_t)$ are defined as follows, the sum of measures is the pointwise sum, and the product of two measures $\mu, \nu \in \mathcal{M}(\mathcal{H}_t)$ is taken as the convolution measure $\mu * \nu (A) := \int \mu(A - x) d\nu(x) \in \mathcal{M}(\mathcal{H}_t)$. In particular, the exponential of a measure $\mu \in \mathcal{M}(\mathcal{H}_t)$ is

$$e^\mu := \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\mu * \cdots * \mu}_{\text{n factors}},$$

which is itself in $\mathcal{M}(\mathcal{H}_t)$. As pointed out in [5], the null term in above sum is the delta measure $\delta_{\{0\}}$ concentrated at the constant path identically equal to zero. This delta measure is the unit element in $\mathcal{M}(\mathcal{H}_t)$, that is, $\mu * \delta_{\{0\}} = \delta_{\{0\}} * \mu = \mu$.

2.3 The Space of Fourier Transform of Measures

Finally, let $\mathcal{F}(\mathcal{H}_t)$ be the Banach algebra of Fourier transforms of measures in $\mathcal{M}(\mathcal{H}_t)$, that is

$$\mathcal{F}(\mathcal{H}_t) := \{\hat{\mu}(\gamma) : \mu \in \mathcal{M}(\mathcal{H}_t)\},$$

where

$$\hat{\mu}(\gamma) := \int_{\mathcal{H}_t} e^{i(\gamma, \eta)} d\mu(\eta).$$

The norm in $\mathcal{F}(\mathcal{H}_t)$ is defined as $\|\hat{\mu}\| := \|\mu\|$. The operations are again the pointwise sum, and the product is the pointwise multiplication $\hat{\mu} \cdot \hat{\nu} = \hat{\mu} * \hat{\nu} \in \mathcal{F}(\mathcal{H}_t)$. In particular, $e^{i\hat{\mu}(\gamma)} = e^{i\hat{\mu}(\gamma)}$.

It is not difficult to see that the Fourier transform operation $\hat{\cdot} : \mathcal{M}(\mathcal{H}_t) \to \mathcal{F}(\mathcal{H}_t)$ defines an isomorphism of Banach algebras.

2.4 A Definition of Path Integral

Following Albeverio and Høegh-Krohn [1], we now define the Feynman path integral of Fresnel type of any functional $f \in \mathcal{F}(\mathcal{H}_t)$ by

$$\int_{\mathcal{H}_t} e^{\frac{i}{\hbar}|\gamma|^2} f(\gamma) d\gamma := \int_{\mathcal{H}_t} e^{-\frac{i}{\hbar}|\gamma|^2} d\mu_f(\gamma),$$

(4)
where \( \mu_f \in \mathcal{M}(\mathcal{H}_t) \) is such that \( \hat{\mu}_f(\gamma) = f(\gamma) \). Observe the integral on the right hand side is a Lebesgue integral respect to the measure \( \mu_f \) which is defined on the space of paths \( \mathcal{H}_t \). Also the integral is well defined since the mapping \( \gamma \mapsto e^{\frac{i}{2} \gamma_3 [\gamma]} \) is a bounded continuous functional on \( \mathcal{H}_t \). This is our definition of path integral and we can only integrate functions on \( \mathcal{H}_t \) which are Fourier transform of bounded measures on \( \mathcal{H}_t \). These functions are called Fresnel integrable functions in [1].

There is also the following useful iterative property of Albeverio and Høegh-Krohn’s path integral of Fresnel type saying that if \( 0 < s < t \) are fixed and \( f, g \) are in \( \mathcal{F}(\mathcal{H}_t) \) such that \( f(\gamma) \) depends only on the values of \( \gamma \) on \( [0, s] \) and \( g(\gamma) \) depends only on the values of \( \gamma \) on \( [s, t] \), and if this is so for any \( \gamma \in \mathcal{H}_t \), then

\[
\int_{\mathcal{H}_t} e^{\frac{i}{2} \gamma_3 [\gamma]} f(\gamma) g(\gamma) d\gamma = \int_{\mathcal{H}_{[s,t]}} e^{\frac{i}{2} \gamma_3 [\gamma]} g(\eta) \int_{\mathcal{H}_s} e^{\frac{i}{2} \gamma_3 [\zeta]} f(\eta + \zeta) d\zeta d\eta,
\]

where \( \mathcal{H}_{[s,t]} \) is the Hilbert space of paths defined on the time interval \( [s, t] \).

We are now ready to state and prove our path integral representation theorem.

3 A Path Integral Representation

**Theorem 1** Let \( \Psi_0(\cdot) \), \( V(t, \cdot) \) and \( K(t, \cdot) \) be in \( \mathcal{F}(\mathbb{R}^d) \) such that \( V(t, x) \) and \( K(t, x) \) are real-valued jointly continuous functions. Then the functional

\[
\gamma \mapsto e^{-i \int_0^t V(s, x + \gamma_s) ds - i \int_0^t K(s, x + \gamma_s) ds} \Psi_0(x + \gamma_0)
\]

(6)

belongs to \( \mathcal{F}(\mathcal{H}_t) \) and

\[
\Psi_t(x) := \int_{\mathcal{H}_t} e^{\frac{i}{2} |\gamma|^2 - i \int_0^t V(s, x + \gamma_s) ds - i \int_0^t K(s, x + \gamma_s) ds} \Psi_0(x + \gamma_0) d\gamma
\]

(7)

is the solution of the stochastic Schrödinger equation

\[
\frac{i}{2} \frac{d \Psi_t(x)}{dt} = \left( -\frac{1}{2} \Delta_x + V(t, x) \right) \Psi_t(x) dt + K(t, x) \Psi_t(x) \circ df_t,
\]

(8)

with initial condition \( \Psi_t(x)|_{t=0} = \Psi_0(x) \).

**Proof.** There are two parts in the proof of this theorem. First we show that the functional (6) belongs to \( \mathcal{F}(\mathcal{H}_t) \) and consequently we can write the
path integral (7). Then we show that (7) does indeed solve equation (8). We start by showing that \( \int_0^t V(s, x + \gamma_s) \, ds \) belongs to \( \mathcal{F}(\mathcal{H}_t) \). Since \( V(s, \cdot) \in \mathcal{F}^0(\mathbb{R}^d) \) we have that for some \( \mu_s \in \mathcal{M}(\mathbb{R}^d) \)

\[
\int_0^t V(s, x + \gamma_s) \, ds = \int_0^t \int_{\mathbb{R}^d} e^{i(y, x + \gamma_s)} \, d\mu_s(y) \, ds \\
= \int_0^t \int_{\mathbb{R}^d} e^{i(y, x)} e^{\iota \sum_k y^k \gamma^k(s)} \, d\mu_s(y) \, ds \\
= \int_0^t \int_{\mathbb{R}^d} e^{i(y, x)} e^{\iota \sum_k (\gamma^k(s), y^k G^k(s, \cdot))} \, d\mu_s(y) \, ds \\
= \int_0^t \int_{\mathcal{H}_t} e^{i(\gamma \xi)} \int_{\mathbb{R}^d} e^{i(y, x)} e^{\iota \sum_k (\gamma^k(s), y^k G^k(s, \cdot))} \, d\mu_s(y) \, ds \\
= \int_0^t \int_{\mathcal{H}_t} e^{i(\gamma \xi)} d \left( \int_0^t \int_{\mathbb{R}^d} e^{i(y, x)} e^{\iota \sum_k (\gamma^k(s), y^k G^k(s, \cdot))} \, d\mu_s(y) \, ds \right) \\
\]

where the underbraced expression is a measure concentrated only on paths \( \xi \) in \( \mathbb{R}^d \) of the form \( \xi(\tau) = y(t + \tau - s) \) with \( y \in \mathbb{R}^d \) and \( s \in [0, t] \). Hence

\[
e^{-i \int_0^t V(s, x + \gamma_s) \, ds} \in \mathcal{F}(\mathcal{H}_t).
\]

A similar argument can be used to prove that \( \Psi_0(x + \gamma_0) \in \mathcal{F}(\mathcal{H}_t) \). Thus, for some \( \sigma \in \mathcal{M}(\mathcal{H}_t) \) we have

\[
e^{-i \int_0^t V(s, x + \gamma_s) \, ds} \Psi_0(x + \gamma_0) = \hat{\sigma}(\gamma).
\]

Analogously we can prove that \( K(s, x + \gamma_s) \in \mathcal{F}(\mathcal{H}_t) \) for every \( s \in [0, t] \). Thus \( K(s, x + \gamma_s) = \hat{\mu}_s(\gamma) \) for some \( \mu_s \in \mathcal{M}(\mathcal{H}_t) \). By hypothesis, since \( s \mapsto \hat{\mu}_s \) is continuous, so is the mapping \( s \mapsto \mu_s \) and hence the measure-valued stochastic integral

\[
\nu_t(\cdot) := \int_0^t \mu_s(\cdot) \, ds
\]

is well defined and belongs to \( \mathcal{M}(\mathcal{H}_t) \). Hence, using our definition (3), we have that

\[
e^{-i \nu_t} \in \mathcal{M}(\mathcal{H}_t).
\]
One can then verify that \( e^{-i \int_0^t K(s,x+\gamma_s) \, ds} \) is the Fourier transform of the measure \( e^{-i \nu_t} \). Indeed

\[
\widehat{e^{-i \nu_t}}(\gamma) = e^{-i \hat{\nu}_t(\gamma)} = e^{-i \int_0^t \hat{\mu}_s(\gamma) \, ds} = e^{-i \int_0^t K(s,x+\gamma_s) \, ds}.
\]

Thus we have shown that

\[
e^{-i \int_0^t V(s,x+\gamma_s) \, ds - i \int_0^t K(s,x+\gamma_s) \, ds} \psi_0(x + \gamma_0) = \sigma \ast e^{-i \nu_t}(\gamma),
\]

and hence the writing out of the path integral (7) is justified.

To prove the second part of our theorem, we first state two preliminary results. Firstly, we observe that the Itô formula applied to the exponential function \( x \mapsto e^{-ix} \) and the measure-valued diffusion process \( \nu_t \) yields the formula

\[
e^{-i \nu_t} = \delta_{\{0\}} - \frac{1}{2} \int_0^t e^{-i \nu_s} \ast \mu_s \ast \mu_s \, d \langle f, f \rangle_s - i \int_0^t e^{-i \nu_s} \ast \mu_s \, df_s. \tag{9}
\]

Here \( \delta_{\{0\}} \) is the delta measure concentrated at the path identically equal to zero and \( \langle f, f \rangle_s \) is the quadratic variation process of the continuous semi-martingale \( f_s \). Secondly, we quote from [1] that the solution of the non-stochastic Schrödinger equation

\[
i \frac{\partial}{\partial t} \psi_t(x) = \left( -\frac{1}{2} \Delta_x + V(t, x) \right) \psi_t(x), \tag{10}
\]

with initial condition \( \psi_t(x)|_{t=0} = \psi_0(x) \) and same conditions on \( V \) and \( \Psi_0 \) as before, can be represented by the path integral

\[
[e^{-iH} \Psi_0](x) = \int_{\mathcal{H}_t} e^{\frac{i}{2} \|\gamma\|^2} - i \int_0^t V(s,x+\gamma_s) \, ds \Psi_0(x + \gamma_0) \, d\gamma,
\]

where \( H = -\frac{1}{2} \Delta_x + V(t, x) \). Observe we are denoting by \( \psi_t(x) \) the solution of the stochastic equation (8) and by \( \psi_t(x) \) the solution of the non-stochastic equation (10). Both equations having the same initial wave function \( \psi_0 \). We now prove that the path integral (7) solves equation (8). By definition and then using Itô formula (9) we have

\[
\psi_t(x) = \int_{\mathcal{H}_t} e^{-\frac{i}{2} \|\gamma\|^2} (\sigma \ast e^{-i \nu_t})(d\gamma)
\]

on 7
\[
\begin{align*}
&= \int_{\mathcal{H}_t} e^{-\frac{i}{2} \gamma^2} (\sigma * (\delta_0 - \frac{1}{2} \int_0^t e^{-i\nu_s} \star \mu_s \star \mu_s \, d\langle f, f \rangle_s - i \int_0^t e^{-i\nu_s} \star \mu_s \, df_s)) \, d\gamma \\
&= \int_{\mathcal{H}_t} e^{-\frac{i}{2} \gamma^2} (\sigma) (d\gamma) + \int_{\mathcal{H}_t} e^{-\frac{i}{2} \gamma^2} (\sigma * (-\frac{1}{2} \int_0^t e^{-i\nu_s} \star \mu_s \star \mu_s \, d\langle f, f \rangle_s)) (d\gamma) \\
&\quad + \int_{\mathcal{H}_t} e^{-\frac{i}{2} \gamma^2} (\sigma * (-i \int_0^t e^{-i\nu_s} \star \mu_s \, df_s)) (d\gamma) \\
&= e^{-itH} \Psi_0(x) - \frac{1}{2} \int_0^t d\langle f, f \rangle_s \int_{\mathcal{H}_t} e^{-\frac{i}{2} \gamma^2} (\sigma * e^{-i\nu_s} \star \mu_s \star \mu_s) (d\gamma) \\
&\quad - i \int_0^t df_s \int_{\mathcal{H}_t} e^{-\frac{i}{2} \gamma^2} (\sigma * e^{-i\nu_s} \star \mu_s) (d\gamma) .
\end{align*}
\]

We now look at the path integral in the last line. We will use the iterative property (5) to find an expression for this integral. Thus, by definition, the referred path integral is

\[
\begin{align*}
&= \int_{\mathcal{H}_t} e^{\frac{i}{2} \gamma^2} \hat{\sigma}(\gamma) e^{-i\nu_s} (\gamma) \hat{\mu}_s(\gamma) \, d\gamma \\
&= \int_{\mathcal{H}_t} e^{\frac{i}{2} \gamma^2} e^{-i \int_0^t V(r,x + \gamma_r) \, dr - i \int_0^t K(r,x + \gamma_r) \, df_r \Psi_0(x + \gamma_0) K(s,x + \gamma_s) \, d\gamma \\
&\quad + \int_{\mathcal{H}_t} e^{\frac{i}{2} \gamma^2} e^{-i \int_0^t V(r,x + \eta_r) \, dr - i \int_0^t K(r,x + \eta_r + \zeta_r) \, df_r \Psi_0(x + \eta_s + \zeta_0) \, d\gamma} d\gamma \\
&= e^{-i(t-s)H} K(s,x) \Psi_s(x) .
\end{align*}
\]

The other path integral is solved similarly. We thus arrive at

\[
\Psi_t(x) = e^{-itH} \Psi_0(x) - \frac{1}{2} \int_0^t e^{-i(t-s)H} K^2(s,x) \Psi_s(x) \, d\langle f, f \rangle_s \\
\quad - i \int_0^t e^{-i(t-s)H} K(s,x) \Psi_s(x) \, df_s .
\]

Multiplying through by \( e^{iH} \) and differentiating respect to \( t \) gives

\[
\begin{align*}
(e^{iH} iH \Psi_t)(x) + (e^{itH} d\Psi_t)(x) &= -\frac{1}{2} e^{itH} K^2(t,x) \Psi_t(x) \, d\langle f, f \rangle_t \\
&\quad - ie^{itH} K(t,x) \Psi_t(x) \, df_t .
\end{align*}
\]
Hence
\[ d\Psi_t(x) = -iH\Psi_t - \frac{1}{2}K^2(t, x)\Psi_t(x)\, df_t - iK(t, x)\Psi_t(x)\, df_t, \]
but this is the Itô version of the Stratonovich equation
\[ id\Psi_t(x) = H\Psi_t + K(t, x)\Psi_t(x)\, df_t. \]
This completes the proof of our theorem.

4 A Stochastic Version of the Mehler Kernel

In this section we specialize to the one dimensional stochastic Schrödinger equation
\[ id\psi_t(x) = \left(-\frac{1}{2} \frac{d^2}{dx^2} + \frac{x^2}{2}\right)\psi_t(x)\, dt + x\psi_t(x)\, df_t, \quad (11) \]
where \( f : [0, \infty) \times \Omega \to \mathbb{R} \) is a continuous semimartingale as before. This equation represents the evolution of a quantum mechanical harmonic oscillator with \( x\psi_t(x)\, df_t(\omega) \) being an added stochastic position operator acting on the wave function \( \psi_t(x) \). When \( f_t = 0 \) the kernel (or propagator) of the above equation is the well known Mehler kernel formula
\[
G_t(x, y) = \frac{1}{\sqrt{2\pi t\sin t}} \exp\left( \frac{i(x^2 + y^2)\cos t - 2xy}{2\sin t} \right).
\]

We have two methods to find the kernel of equation (11). One method is by using a discretisation of the path integral and a second method is via a change of variables. These two methods are explained in the following two sections. In either case we make use of the path integral representation of the kernel in configuration space stated in our next theorem. The potential functions \( V(t, x) = \frac{1}{2}x^2 \) and \( K(t, x) = x \) corresponding to the equation (11), are not elements of \( \mathcal{F}(\mathbb{R}^d) \), so we cannot apply our representation theorem. However, the expression of the path integral suggested by our theorem, gives the right solution as we will see.

The approach we follow is the following. We take the path integral as an ansatz, then we solve the path integral and find the kernel. It can then be checked that the found kernel solves the Schrödinger equation (11).

**Theorem 2** The kernel \( G_t(x, y) \) for the stochastic Schrödinger equation
\[ id\psi_t(x) = \left(-\frac{1}{2} \frac{d^2}{dx^2} + \frac{x^2}{2}\right)\psi_t(x)\, dt + x\psi_t(x)\, df_t, \quad (12) \]
with initial condition $\Psi_0(x) = \delta_y(x)$, has the path integral representation

$$G_t(x, y) = \int_{\mathcal{H}_t} e^{iS(x+q)} \delta_y(x + q_0) \, dq,$$  \hspace{1cm} (13)

where the action $S$ is given by

$$S(x + q) = \frac{1}{2} \int_0^t |\dot{q}_s|^2 \, ds - \frac{1}{2} \int_0^t |x + q_s|^2 \, ds - \int_0^t (x + q_s) \circ df_s.$$  \hspace{1cm} (14)

This gives the kernel

$$G_t(x, y) = \frac{1}{\sqrt{2\pi i \sin t}} \exp \left( \frac{i}{2 \sin t} \left( x^2 + y^2 \cos t - 2xy \right) \right) \exp \left( i \int_0^t -x \sin r + y \sin(r - t) \circ df_r \right) \exp \left( i \int_0^t \left[ \int_0^r \sin s \sin(r - t) \circ df_s \right] \circ df_r \right).$$  \hspace{1cm} (15)

We observe the above formula holds only for $t \neq k\pi$, and the first line of it is the classical Mehler kernel formula. The following two sections contain two different ways to prove this result.

### 4.1 Solution Via Discretisation

We here discretise the path integral (13), and obtain solution (15). As expected, this method involves lengthy calculations. Let $0 < t_1 < t_2 < \cdots < t_N < t$ be a partition of the time interval $[0, t]$ into $N + 1$ subintervals such that $t_{i+1} - t_i = \epsilon > 0$, and $(N + 1)\epsilon = t$. Then the action $S$ can be approximated by

$$S(\tilde{q}) = \frac{1}{2} \left[ \sum_{j=0}^{N} (q_{j+1} - q_j)^2 - \sum_{j=0}^{N} (q_j + q_{j+1})^2 - \sum_{j=0}^{N} (q_j + q_{j+1}) \Delta_j f \right],$$

where $\tilde{q}$ stands for the vector $(q_1, \ldots, q_N)$, with $q_j = q(t_j)$, $\Delta_j f = f(t_{j+1}) - f(t_j)$, $q_0 = y$, and $q_t = x$.

The path integral is then assumed to be calculated as follows

$$G_t(x, y) = \lim_{\epsilon \to 0} \frac{(2\pi i \epsilon)^{-(N+1)/2}}{\mathbb{R}^N} \int e^{iS(\tilde{q})} \, d\tilde{q},$$
where the limit is taken in such a way that also $N \to \infty$, and $(N+1)e = t$ 
remains constant. Let us now define

$$
a := \frac{2}{\epsilon} - \frac{\epsilon}{2} ,
$$

$$
b := \frac{1}{\epsilon} + \frac{\epsilon}{4} ,
$$

and

$$
C := \frac{a}{2}(x^2 + y^2) + x(-\Delta_N f) + y(-\Delta_0 f) .
$$

Then the discretised action $S(\tilde{q})$ can be written as

$$
S(\tilde{q}) = \frac{1}{2}(\tilde{q}^TA\tilde{q} + B^T\tilde{q} + C) ,
$$

where $A$ and $B$ are the following matrix and vector.

$$
A = \begin{pmatrix}
a & -b & 0 & 0 & \cdots \\
-b & a & -b & 0 & \cdots \\
0 & -b & a & -b & \cdots \\
0 & 0 & -b & a & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}_{N \times N}
$$

$$
B = \begin{pmatrix}
-\Delta_0 f - \Delta_1 f - 2yb \\
-\Delta_1 f - \Delta_2 f \\
\vdots \\
-\Delta_{N-2} f - \Delta_{N-1} f \\
-\Delta_{N-1} f - \Delta_N f - 2xb \\
\end{pmatrix}_{N \times 1}
$$

Since $A$ is a real symmetric matrix we have the formula

$$
\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}(q^TAq+B^Tq+C)}d\tilde{q} = (2\pi\hbar)^{N/2} |A|^{-1/2}e^{\frac{i}{\hbar}(C-\frac{1}{4}B^TA^{-1}B)} ,
$$

and thus we obtain the expression

$$
G_t(x,y) = \lim_{\epsilon \to 0} (2\pi\hbar)^{-1/2} e^{-(N+1)/2} |A|^{-1/2}e^{\frac{i}{\hbar}(C-\frac{1}{4}B^TA^{-1}B)} .
$$

Now we write $B^T = B_1^T + B_2^T$, and $C = C_1 + C_2$ where

$$
B_1^T := ( -2yb, 0, \ldots, 0, -2xb ) ,
$$

$$
B_2^T := ( \Delta_0 f - \Delta_1 f, \Delta_1 f - \Delta_2 f, \ldots, -\Delta_{N-1} f - \Delta_N f ) ,
$$

$$
C_1 := \frac{a}{2}(x^2 + y^2) ,
$$

$$
C_2 := x(-\Delta_N f) + y(-\Delta_0 f) .
$$

Then $B^T A^{-1} B^T = ( B_1^T A^{-1} B_1 + B_1^T A^{-1} B_2 + B_2^T A^{-1} B_1 + B_2^T A^{-1} B_2 )$, 

and hence we can express $G_t(x,y)$ as the following product of limits

$$
G_t(x,y) = \lim_{\epsilon \to 0} (2\pi\hbar)^{-1/2} e^{-(N+1)/2} |A|^{-1/2}
$$

$$
\times \lim_{\epsilon \to 0} e^{\frac{i}{\hbar}(C_1-\frac{1}{4}B_1^T A_1^{-1} B_1)} e^{\frac{i}{\hbar}(C_2-\frac{1}{4}B_2^T A_2^{-1} B_2)} .
$$

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\[
\lim_\epsilon \exp\left[ \frac{i}{2} (C_1 - \frac{1}{4} B_1^T A^{-1} B_1) \right] \\
\lim_\epsilon \exp\left[ \frac{i}{2} (C_2 - \frac{1}{4} B_2^T A^{-1} B_1) \right] \\
\lim_\epsilon \exp\left[ \frac{i}{2} (-\frac{1}{4} B_1^T A^{-1} B_2) \right] \\
\lim_\epsilon \exp\left[ \frac{i}{2} (-\frac{1}{4} B_2^T A^{-1} B_2) \right].
\]

The difficult part in this method lies in the calculation of \(|A|\) and \(A^{-1}\). See [4] for the details of these calculations. We can then show that all the above limits exist, and that they are in fact given by the following expressions.

(a) \( \lim_{\epsilon \to 0} (2\pi \epsilon)^{-1/2} e^{-(N+1)/2} |A|^{-1/2} = (2\pi i \sin t)^{-1/2}. \)

(b) \( \lim_{\epsilon \to 0} \exp\left[ \frac{i}{2} (C_1 - \frac{1}{4} B_1^T A^{-1} B_1) \right] = \exp\left( \frac{i}{2} \int_0^t -x \sin r + y \sin(r - t) \, df_r \right). \)

(c) \( \lim_{\epsilon \to 0} \exp\left[ \frac{i}{2} (C_2 - \frac{1}{4} B_2^T A^{-1} B_1) \right] = \exp\left( \frac{i}{2} \int_0^t -x \sin r + y \sin(r - t) \, df_r \right). \)

(d) \( \lim_{\epsilon \to 0} \exp\left[ \frac{i}{2} (-\frac{1}{4} B_1^T A^{-1} B_2) \right] = \exp\left( \frac{i}{2} \int_0^t -x \sin r + y \sin(r - t) \, df_r \right). \)

(e) \( \lim_{\epsilon \to 0} \exp\left[ \frac{i}{2} (-\frac{1}{4} B_2^T A^{-1} B_2) \right] = \exp\left( \frac{i}{2} \int_0^t -x \sin r + y \sin(r - t) \, df_r \right). \)

Collecting all those limits together gives solution (15). Once this solution is obtained, we can directly substitute it in our Schrödinger equation (12) and check that it is indeed a fundamental solution.

### 4.2 Solution Using A Change of Variables

Here is a more straightforward but formal way to solve the equation (12) by using a change of path in our path integral. We start again with our configuration space path integral representation

\[
G_t(x, y) = \int_{\mathcal{H}_t} e^{\frac{i}{2} \int_0^t \dot{q}_s^2 ds - \frac{i}{2} \int_0^t (x + q_s)^2 ds - i \int_0^t (x + q_s) \, df_s} \delta_y(x + q_0) \, dq.
\]

Through a change of variables we will reduce the above path integral to the case of the classical Mehler kernel formula. We define the translation \( T : \mathcal{H}_t \to \mathcal{H}_t \) given by

\[
Tq_s = q_s + \xi_s,
\]
where the path $\xi : [0, t] \to \mathbb{R}$ is the solution of the stochastic integro-differential equation

\[
d\xi_s + \xi_s \, ds = \circ d f_s ,
\]

(17)

for $s \in [0, t]$, subject to the boundary conditions $\xi_t = 0$, and $\dot{\xi}_0 = 0$. The stochastic equation (17) can be solved explicitly, its solution can be directly checked to be

\[
\xi(s) = \frac{\cos s}{\cos t} \int_0^t \sin(u - t) \circ d f_u + \int_0^s \sin(s - u) \circ d f_u .
\]

(18)

Then, under this translation, our path integral representation can be written as

\[
G_t(x, y) = \int_{\mathcal{H}_t} e^{\frac{i}{2} \int_0^t (\hat{q}_s^2 - (x + q_s)^2) \, ds - i \int_0^t (x + q_s - \xi_s) \circ d f_s} \delta_y(x + q_0 - \xi_0) \, dq .
\]

Expanding out the squares, the exponent in our last expression can be written as follows

\[
\begin{align*}
&\frac{i}{2} \int_0^t (\hat{q}_s^2 - (x + q_s)^2) \, ds \\
&+ i \int_0^t (q_s \dot{\xi}_s + q_s \xi_s) \, ds - i \int_0^t q_s \circ d f_s \\
&+ i \int_0^t \left( \frac{1}{2} \ddot{\xi}_s^2 - \frac{1}{2} \xi_s^2 + x \xi_s \right) \, ds - i \int_0^t (x - \xi_s) \circ d f_s .
\end{align*}
\]

The first line of the above expression we will keep inside in path integral. The second line is obtained by observing that

\[
\int_0^t \dot{q}_s \dot{\xi}_s \, ds = - \int_0^t q_s \dddot{\xi}_s \, ds ,
\]

where the boundary conditions for $\xi_s$ are used. This second line vanishes by using the differential equation for $\xi_s$, multiplied by $q_s$ and integrated. The third and last line we will place outside the path integral. This then gives us the expression

\[
G_t(x, y) = e^{\frac{i}{2} \int_0^t \left( \frac{1}{2} \dot{\xi}_s^2 - \frac{1}{2} \xi_s^2 + x \xi_s \right) \, ds - i \int_0^t (x - \xi_s) \circ d f_s} \\
\cdot \int_{\mathcal{H}_t} e^{\frac{i}{2} \int_0^t \hat{q}_s^2 \, ds - i \int_0^t \left( x + q_s \right)^2 \, ds} \delta_y(x + q_0) \, dq ,
\]

where the last path integral is solved by the Mehler kernel formula

\[
(2\pi\sin t)^{-1/2} \exp \left\{ i \frac{[x^2 + (y + \xi_0)^2] \cos t - 2x(y + \xi_0)}{2\sin t} \right\} .
\]
Comparing the above solution with (15), we see that the coefficients of the terms \(x^2\), \(y^2\), and \(-2xy\) in the exponent, are the correct ones. We then need to verify the coefficients corresponding to the terms \(y\), \(x\), and the independent term, to completely verify our solution. To do this we need from our solution (18)

\[
\xi_0 = \frac{1}{\cos t} \int_0^t \sin (u - t) \circ df_u ,
\]

and

\[
\dot{\xi}_0 = -\frac{\sin s}{\sin t} \int_0^t \sin (u - t) \circ df_u + \int_0^s \cos (u - s) \circ df_u .
\]

With these formulae we can easily prove the following identities. For the coefficient of \(y\):

\[
\frac{\xi_0 \cos t}{\sin t} = \int_0^t \frac{\sin r - t}{\sin t} \circ df_r .
\]

For the coefficient of \(x\):

\[
-\frac{\xi_0}{\sin t} + \int_0^t \xi_s \circ ds - \int_0^t \circ df_s = -\frac{1}{\sin t} \int_0^t \sin r \circ df_r .
\]

For the independent term:

\[
\frac{\xi_0 \cos t}{2\sin t} + \frac{1}{2} \int_0^t \left( \frac{\xi_s^2 - \xi_s^2}{\sin t} \right) ds + \int_0^t \xi_s \circ df_s = \int_0^t \left[ \int_0^t \frac{\sin s \sin (r - t)}{\sin t} \circ df_s \right] \circ df_r .
\]

The proofs of these three identities are straightforward but require some lengthy computations. After this we can then reconstruct our solution (15). This finishes our work. In a forthcoming paper we will study path integral representations of the solution of certain stochastic Schrödinger equations on phase space.

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References


